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A generalization of the restricted isometry property and applications to compressed sensing

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Abstract. This paper introduces a new general theory of compressed sensing. We give a natural generalization of the restricted isometry property (RIP) called weak RIP. We consider the proposed theory to be more useful for real data analysis than the RIP. In this note, we verify the accuracy of the weak RIP by showing in reconstruction from undersampling measurements where it is possible to improve estimation in various situations.

Keywords. compressed sensing, restricted isometry constants, restricted isometry property, sparse approximation, sparse signal recovery, weak restricted isometry property

1. Introduction

1.1. RIP

This paper introduces the theory of compressed sensing (CS). CS theory asserts that one can recover certain signals and images from only a few samples or measurements. Here, we consider

$$\boldsymbol{y} = A\boldsymbol{x}, \qquad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{1}$$

where A is an $m \times n$ matrix. Our goal is to reconstruct $\boldsymbol{x} \in \boldsymbol{R}^n$ with good accuracy. We are interested in the ill-posed problem when $m < n$. It is known that when $\boldsymbol{x}$ is sparse, or approximately sparse, and A obeys the restricted isometry property (RIP), one can accurately reconstruct $\boldsymbol{x}$ from the measurements $\boldsymbol{y} = A\boldsymbol{x}$. In fact, the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \|\tilde{\boldsymbol{x}}\|_1 \quad \text{subject to} \quad \boldsymbol{y} = A\tilde{\boldsymbol{x}} \tag{2}$$

recovers $\boldsymbol{x}$ exactly, where $\|\cdot\|_1$ is the l_1 norm. Furthermore, we extend this method to noisy recovery. Suppose we observe

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{3}$$

where $\boldsymbol{z}$ is an unknown noise term. In this context, we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\tilde{\boldsymbol{x}} \in \boldsymbol{R}^n} \|\tilde{\boldsymbol{x}}\|_1 \quad \text{subject to} \quad \|\boldsymbol{y} - A\tilde{\boldsymbol{x}}\|_2 \le \varepsilon, \tag{4}$$

where ε is an upper bound on the size of the noisy contribution and $\|\cdot\|_2$ is the l_2 norm.

Definition 1. A matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1-\delta)\|\boldsymbol{a}\|_2^2 \le \|A\boldsymbol{a}\|_2^2 \le (1+\delta)\|\boldsymbol{a}\|_2^2 \tag{5}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be s-sparse if it has at most s nonzero entries. The minimum of the above constants δ is said to be the isometry constant of A and is denoted by δ_s.

The condition (5) is equivalent to requiring that the matrix $A_S^T A_S$ has all of its eigenvalues in $[1-\delta_s, 1+\delta_s]$, where A_S is the $m \times |S|$ matrix composed of these columns for any subset S of $\{1, 2, \ldots, n\}$. Here $|S|$ is number of elements of S. It has been shown that l_1 optimization can recover an unknown signal in noiseless case and noisy case under various sufficient conditions on δ_s or δ_{2s}. For example, E. J. Candès and T. Tao [4] have proved that if $\delta_{2s} < \sqrt{2}-1$, then an unknown signal can be recovered. Later, S. Foucart and M. Lai [8] have improved the bound to $\delta_{2s} < 0.4531$. Others, $\delta_{2s} < 0.4652$ is used by S. Foucart [7], $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large by T. Cai et al. [1], $\delta_{2s} < 0.4734$ for the case such that s is very large by S. Foucart [7] and $\delta_s < 0.307$ by T. Cai et al. [1]. In a resent paper, Q. Mo and S. Li [10] have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ in some special case. Recently H. Inoue [9] has improved the sufficient condition to $\delta_s < 0.309$ for general case and $\delta_s < 0.472$ and $\delta_{2s} < 0.593$ in some special case.

1.2. Weak RIP and main theorems

This paper shows that it is possible to apply CS theory to various fields. For example, when we apply CS to a statistical model, we define A as a basis function matrix and $\boldsymbol{x}$ as a coefficient vector. We have to estimate the coefficient vector and assess this model. In this case, if A is a random matrix, we can not interpret the estimated model. Thus, in order to interpret models, it is important to discuss the method of using a matrix according to the structure of the data and the assessment of estimators. However, the RIP requires a bounded condition number for all submatrices built by selecting s arbitrary columns and the spectral norm of a matrix is generally difficult to calculate. Therefore, it seems useful to weaken the condition of RIP. E. J. Candès and Y. Plan [3] have introduced the notion of weak RIP which is a generalization of RIP as follows:

Definition 2 (Weak RIP)**.** Let $T_0 \subset \{1, 2, \ldots, n\}$ with $|T_0| = s$ and $1 < r < s$. A obeys the weak RIP with respect to T_0 of order r if there exists $0 < \delta < 1$ such that for any subset $R \subset T_0^c$ with $|R| \leq r$,

$$(1-\delta)\,\|\boldsymbol{x}_{T_0\cup R}\|_2^2 \leq \|A\boldsymbol{x}_{T_0\cup R}\|_2^2 \leq (1+\delta)\,\|\boldsymbol{x}_{T_0\cup R}\|_2^2 \quad (6)$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$. The minimum of such constants δ is denoted by $\delta_{T_0,r}$.

Roughly speaking the notion of the weak RIP, we choose a suitable location T_0 with $|T_0| = s$ in the columns of the matrix A. We remark that A obeys the RIP of order r, but it does not necessarily obey the RIP of order $(s + r)$. Furthermore, the matrix $A_{T_0\cup R}$ obeys the inequality (6) for any subset R of T_0^c with $|R| = r$. In [3] they have proved that under the assumptions of isotropy property and incoherence property a random matrix obeys the weak RIP with high probability $1 - 5e^{-\beta}$ if $m \geq C \log n$ (where C is a constant which only depends on β, δ, s, r and the coherent parameter μ), and have evaluated stochastically the solution of LASSO [11] using the weak RIP and the other properties (the existence of inexact dual vector, the noise correlation bound and etc.) In this paper, we focus on this notion and evaluate the solution of CS under the assumption of only the weak RIP without the probability, and obtain almost the same results (the following Theorem 1 and Theorem 2) as for the case of the RIP. In case that we have some information about the data, that is, we have a good location T_0, it seems better to analyze data using the weak RIP because it is much easier to construct matrices obeying the weak RIP than matrices obeying the RIP.

Throughout this section, let A be an $m \times n$ matrix. For matrix A and a subset $T \subset \{1, 2, \ldots, n\}$, A_T denotes the $m \times |T|$ matrix with column indices in T. Also, $A_{\{i\}}$ is the i-th column of A. Likewise, for a vector $\boldsymbol{a} \in \boldsymbol{R}^n$, $\boldsymbol{a}_T$ is the restriction of $\boldsymbol{a}$ to indices in T. Thus, if $\boldsymbol{a}$ is supported on T, $A\boldsymbol{a} = A_T\boldsymbol{a}_T$. Furthermore, the identity matrix, in any dimension, is denoted I, and the operator norm of A is denoted $\|A\|$.

Theorem 1. *Let T_0 be a fixed subset of $\{1, 2, \ldots, n\}$. Assume that A obeys the weak RIP with respect to T_0 of order r, and $\delta_{T_0.r} < \frac{1}{1+\sqrt{s/[r/2]}}$, where $[\cdot]$ is the floor function. Then, the solution $\boldsymbol{x}^\star$ to (4) obeys*

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq D_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + D_1\varepsilon, \quad (7)$$

where

$$D_0 = \frac{2}{\sqrt{[r/2]}\big(1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta_{T_0,r}\big)},$$

$$D_1 = \left(1 + \sqrt{\frac{s}{[r/2]}}\right)\frac{2\sqrt{1+\delta_{T_0,r}}}{\big(1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta_{T_0,r}\big)}.$$

In particular, if $\boldsymbol{x}$ is a T_0-sparse vector, then $\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \leq D_1\varepsilon$.

The following theorem is only a slight generalization of Theorem 1; however, it is useful in order to construct a matrix A obeying the weak RIP (see Section 3 for examples).

Theorem 2. *Let T_0 be a subset of $\{1, 2, \ldots, n\}$ with $|T_0| = s$, and r be a natural number such that $1 < r < s$. Suppose that*

(i) *$\{A_{\{i\}};\ i \in T_0 \cup R\}$ is linearly independent for any subset R of T_0^c with $|R| = r$;*

(ii) *$\delta_0 \equiv 1 - \left(\frac{\min_R c_R}{\max_R \|A_{T_0\cup R}\|}\right)^2 < \frac{1}{1+\frac{1}{2}\sqrt{s/[r/2]}}$, where $c_R = \max\{c > 0;\ c\|\boldsymbol{x}_{T_0\cup R}\|_2^2 \leq \|A\boldsymbol{x}_{T_0\cup R}\|_2^2,\ \boldsymbol{x} \in \boldsymbol{R}^n\}$.*

Then,

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq E_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + E_1\varepsilon,$$

where

$$E_0 = \frac{1}{\sqrt{[r/2]}}\left(\frac{2-\delta_0}{1 - \big(1 + \frac{1}{2}\sqrt{s/[r/2]}\,\big)\delta_0}\right),$$

$$E_1 = \frac{2\big(1 + \sqrt{s/[r/2]}\,\big)}{\max_R \|A_{T_0\cup R}\|\big(1 - \big(1 + \frac{1}{2}\sqrt{s/[r/2]}\,\big)\big)\delta_0}.$$

2. Proofs

In this section, we prove Theorem 1 and Theorem 2. Here, we simply set $\delta = \delta_{T_0\cup R}$. By Definition 1, we have the following:

Lemma 1. *Take arbitrary subsets R_1 and R_2 of T_0^c such that $R_1 \cap R_2 = \emptyset$ and $|R_1| + |R_2| \leq r$. Then,*

$$|\langle A\boldsymbol{a}, A\boldsymbol{b}\rangle| \leq \delta\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2 \quad (8)$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$ with $\operatorname{supp}\boldsymbol{a} \subset T_0 \cup R_1$ and $\operatorname{supp}\boldsymbol{b} \subset R_2$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

Proof. Take arbitrary $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$ with $\operatorname{supp}\boldsymbol{a} \subset T_0 \cup R_1$ and $\operatorname{supp}\boldsymbol{b} \subset R_2$. We may suppose $\|\boldsymbol{a}\|_2 = \|\boldsymbol{b}\|_2 = 1$ without loss of generality to show the inequality (8). We put $R = R_1 \cup R_2$. Since A obeys the weak RIP with respect to T_0 of order r and $\boldsymbol{a}$ is orthogonal to $\boldsymbol{b}$, it follows that

$$\begin{aligned} 4\,\langle A\boldsymbol{a}, A\boldsymbol{b}\rangle &= \|A_{T_0\cup R}(\boldsymbol{a}+\boldsymbol{b})\|_2^2 - \|A_{T_0\cup R}(\boldsymbol{a}-\boldsymbol{b})\|_2^2 \\ &\leq (1+\delta)\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 - (1-\delta)\|\boldsymbol{a}-\boldsymbol{b}\|_2^2 \\ &= 4\delta \end{aligned}$$

and

$$\begin{aligned} 4\,\langle A\boldsymbol{a}, A\boldsymbol{b}\rangle &\geq (1-\delta)\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 - (1+\delta)\|\boldsymbol{a}-\boldsymbol{b}\|_2^2 \\ &= -4\delta, \end{aligned}$$

which implies that

$$|\langle A\boldsymbol{a}, A\boldsymbol{b}\rangle| \leq \delta. \qquad \square$$

Proof of Theorem 1. We set $\boldsymbol{h} = \boldsymbol{x}^\star - \boldsymbol{x}$. By the linearity of A and the triangle equality, we have

$$\|A\boldsymbol{h}\|_2 \leq 2\varepsilon. \tag{9}$$

Let T_1 be the locations of the $[r/2]$ largest coefficients of $\boldsymbol{h}_{T_0^c}$. Repeating this method, $\{1, 2, \ldots, n\}$ is decomposed as $\{1, 2, \ldots, n\} = T_0 \cup T_1 \cup \cdots \cup T_{l-1} \cup T_l$, $|T_l| \leq [r/2]$. We may assume without loosing generality that

$$\begin{aligned} \boldsymbol{h}_{T_0} &= (h_1^{(T_0)}, h_2^{(T_0)}, \ldots, h_s^{(T_0)}, 0, \ldots, 0) \\ \boldsymbol{h}_{T_1} &= (0, \ldots, 0, h_1^{(T_1)}, \ldots, h_{[r/2]}^{(T_1)}, 0, \ldots, 0) \\ &\vdots \\ \boldsymbol{h}_{T_{l-1}} &= (0, \ldots, 0, h_1^{(T_{l-1})}, \ldots, h_{[r/2]}^{(T_{l-1})}, 0, \ldots, 0) \\ \boldsymbol{h}_{T_l} &= (0, \ldots 0, h_1^{(T_l)}, \ldots, h_{|T_l|}^{(T_l)}). \end{aligned}$$

Then, since

$$\left|h_k^{(T_{j-1})}\right| \geq \max_{k \in T_j} \left|h_k^{(T_j)}\right|, \quad 2 \leq j \leq l,\ 1 \leq k \leq \left[\frac{r}{2}\right],$$

it follows that for any j such that $2 \leq j \leq l$,

$$\left\|\boldsymbol{h}_{T_j}\right\|_2 \leq \frac{1}{\sqrt{[r/2]}} \left\|\boldsymbol{h}_{T_{j-1}}\right\|_1, \tag{10}$$

which implies that

$$\left\|\boldsymbol{h}_{(T_0 \cup T_1)^c}\right\|_2 = \left\|\sum_{j \geq 2} \boldsymbol{h}_{T_j}\right\|_2 \leq \frac{1}{\sqrt{[r/2]}} \|\boldsymbol{h}_{T_0^c}\|_1. \tag{11}$$

Since

$$\begin{aligned} \|\boldsymbol{x}\|_1 &\geq \|\boldsymbol{x}^\star\|_1 \\ &= \|\boldsymbol{x}_{T_0} + \boldsymbol{h}_{T_0} + \boldsymbol{x}_{T_0^c} + \boldsymbol{h}_{T_0^c}\|_1 \\ &\geq \|\boldsymbol{x}_{T_0}\|_1 - \|\boldsymbol{h}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0^c}\|_1 - \|\boldsymbol{x}_{T_0^c}\|_1, \end{aligned}$$

it follows that

$$\|\boldsymbol{h}_{T_0^c}\|_1 \leq 2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0}\|_1, \tag{12}$$

which implies by (11) and the Schwartz inequality

$$\begin{aligned} \|\boldsymbol{h}_{(T_0 \cup T_1)^c}\|_2 &\leq \frac{1}{\sqrt{[r/2]}} \|\boldsymbol{h}_{T_0}\|_1 + \frac{2}{\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\ &\leq \sqrt{\frac{s}{[r/2]}} \|\boldsymbol{h}_{T_0}\|_2 + \frac{2}{\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\ &\leq \sqrt{\frac{s}{[r/2]}} \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 + \frac{2}{\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1. \end{aligned} \tag{13}$$

Furthermore, it follows from Lemma 1 that

$$\left|\left\langle A\boldsymbol{h}_{T_0 \cup T_1}, A\boldsymbol{h}_{T_j}\right\rangle\right| \leq \delta \|\boldsymbol{h}_{T_j}\|_2 \|\boldsymbol{h}_{T_0 \cup T_1}\|_2,$$

which implies by (9) and (10) that for any $j \geq 2$,

$$\begin{aligned} \|A\boldsymbol{h}_{T_0 \cup T_1}\|_2^2 &= \left\langle A\boldsymbol{h}_{T_0 \cup T_1}, A\boldsymbol{h} - \sum_{j \geq 2} A\boldsymbol{h}_{T_j}\right\rangle \\ &\leq \|A\boldsymbol{h}_{T_0 \cup T_1}\|_2 \|A\boldsymbol{h}\|_2 + \sum_{j \geq 2} \left|\langle A\boldsymbol{h}_{T_0 \cup T_1}, A\boldsymbol{h}_{T_j}\rangle\right| \qquad (14) \\ &\leq \sqrt{1 + \delta}\, \|\boldsymbol{h}_{T_0 \cup T_1}\|_2\, 2\varepsilon + \delta \left(\sum_{j \geq 2} \|\boldsymbol{h}_{T_j}\|_2\right) \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 \\ &\leq \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 \left(2\varepsilon\sqrt{1 + \delta} + \delta \frac{1}{\sqrt{[r/2]}} \|\boldsymbol{h}_{T_0^c}\|_1\right). \end{aligned}$$

Hence, it follows from (6) and (12) that

$$\begin{aligned} &(1 - \delta)\|\boldsymbol{h}_{T_0 \cup T_1}\|_2 \\ &\quad \leq 2\varepsilon\sqrt{1 + \delta} + \frac{\delta}{\sqrt{[r/2]}} (2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \sqrt{s}\, \|\boldsymbol{h}_{T_0 \cup T_1}\|_2), \end{aligned}$$

so that

$$\begin{aligned} &\left(1 - \left(1 + \sqrt{\frac{s}{[r/2]}}\right)\delta\right) \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 \\ &\qquad\qquad \leq 2\varepsilon\sqrt{1 + \delta} + \frac{2\delta}{\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1. \end{aligned} \tag{15}$$

By assumption: $\delta < \frac{1}{1 + \sqrt{s/[r/2]}}$ (if and only if, $1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta > 0$), we have

$$\begin{aligned} \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 &\leq \frac{2\sqrt{1 + \delta}}{1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta} \varepsilon \\ &\quad + \frac{2\delta}{\big(1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta\big)\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1. \end{aligned} \tag{16}$$

Thus, we have by (13) and (16)

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 &= \|\boldsymbol{h}\|_2 \leq \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 + \|\boldsymbol{h}_{(T_0 \cup T_1)^c}\|_2 \\ &\leq \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 + \sqrt{\frac{s}{[r/2]}} \|\boldsymbol{h}_{T_0 \cup T_1}\|_2 + \frac{2}{\sqrt{[r/2]}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\ &\leq \left(1 + \sqrt{\frac{s}{[r/2]}}\right) \frac{2\sqrt{1 + \delta}}{\big(1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta\big)} \varepsilon \\ &\quad + \frac{2}{\sqrt{[r/2]}\big(1 - \big(1 + \sqrt{s/[r/2]}\,\big)\delta\big)} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1. \end{aligned} \tag{17}$$

This completes the proof. $\square$

Proof of Theorem 2. Take an arbitrary $R \subset T_0^c$ with $|R| = r$. By (i), the operator $A_{T_0 \cup R}$ is injective, and so there exists a positive constant $c > 0$ such that

$$c\|\boldsymbol{x}_{T_0 \cup R}\|_2 \leq \|A_{T_0 \cup R}\boldsymbol{x}_{T_0 \cup R}\|_2$$

for each $\boldsymbol{x} \in \boldsymbol{R}^n$. We set $c_R = \max\{c > 0;\ c\|\boldsymbol{x}_{T_0 \cup R}\|_2^2 \leq \|A\boldsymbol{x}_{T_0 \cup R}\|_2^2,\ \boldsymbol{x} \in \boldsymbol{R}^n\}$. Then,

$$\begin{aligned} \left(\min_R c_R\right) \|\boldsymbol{x}_{T_0 \cup R}\|_2 &\leq \|A\boldsymbol{x}_{T_0 \cup R}\|_2 \\ &\leq \left(\max_R \|A_{T_0 \cup R}\|\right) \|\boldsymbol{x}_{T_0 \cup R}\|_2 \end{aligned} \tag{18}$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$. Here, we put $\delta_0 = 1 - \left(\frac{\min_R c_R}{\max_R \|A_{T_0\cup R}\|}\right)^2$. By (18), we have

$$\left(\max_R \|A_{T_0\cup R}\|^2\right)(1-\delta_0)\,\|\boldsymbol{x}_{T_0\cup R}\|_2^2 \leq \|A\boldsymbol{x}_{T_0\cup R}\|_2^2 \leq \left(\max_R \|A_{T_0\cup R}\|^2\right)\|\boldsymbol{x}_{T_0\cup R}\|_2^2 \quad (19)$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$, which implies that

$$|\langle A\boldsymbol{h}_{T_0\cup T_1}, A\boldsymbol{h}_{T_j}\rangle| \leq \frac{(\max_R \|A_{T_0\cup R}\|^2)\delta_0}{2}\|\boldsymbol{h}_{T_j}\|_2\|\boldsymbol{h}_{T_0\cup T_1}\|_2 \quad (20)$$

for all $j \geq 2$. Using (19) and (20), we can prove Theorem 2 in the same way as Theorem 1. □

3. Examples

In this section, we give simple examples of $m \times n$ matrices obeying the weak RIP.

Example 1. Let T_0 be a subset of $\{1, 2, \ldots, n\}$ with $|T_0| = s$ and r be a natural number with $0 < r < s$. Suppose that A_{T_0} and $A_{T_0^c}$ satisfy independently the following (1a) and (1b), respectively:

(1a) A_{T_0} obeys the RIP of order s. We denote by δ_{T_0} the isometry constant of A_{T_0}.

(1b) $A_{T_0^c}$ obeys the RIP of order r. We denote by δ_r the isometry constant of $A_{T_0^c}$.

Furthermore, suppose that A_{T_0} and $A_{T_0^c}$ have the following relation (1c):

(1c) A obeys the mutual incoherence property (MIP), which requires that the maximum pairwise correlation of columns of $(A_{T_0}, A_{T_0}^c)$ is small, i.e., $|\langle A_{\{i\}}, A_{\{j\}}\rangle| < \varepsilon$ for any $i \in T_0$ and $j \in T_0^c$, where $0 < \varepsilon < \frac{1-\max(\delta_{T_0},\delta_r)}{sr}$.

Many researchers have studied about the MIP. For example, D. L. Donoho and X. Huo [6] have introduced the property of the MIP and T. Cai et al. [2] have introduced the connections between the RIP and the MIP.

Then the inequality (6) holds for all matrices $A_{T_0\cup R}$ constructed by mixed locations $T_0 \cup R$ of T_0 and T_0^c, which means that A obeys the weak RIP with respect to T_0 of order r. Indeed, take arbitrary $R = \{n_1', n_2', \ldots, n_r'\} \subset T_0^c$ and $\boldsymbol{x} \in \boldsymbol{R}^n$. Since

$$\begin{aligned}|\langle A_{T_0}^\star A_R\boldsymbol{x}_R, \boldsymbol{x}_{T_0}\rangle| &\leq \sum_{j=1}^r\sum_{i=1}^s |\langle A_{\{n_i\}}, A_{\{n_j'\}}\rangle|\,|x_{n_i}x_{n_j'}| \\ &\leq sr\varepsilon\|\boldsymbol{x}_{T_0}\|_2\|\boldsymbol{x}_R\|_2,\end{aligned}$$

it follows that

$$\begin{aligned}&|\langle (A_{T_0\cup T_R}^\star A_{T_0\cup R} - I)\boldsymbol{x}, \boldsymbol{x}\rangle| \\ &\quad= |\langle (A_{T_0}^\star A_{T_0} - I)\boldsymbol{x}_{T_0}, \boldsymbol{x}_{T_0}\rangle| \\ &\qquad + 2|\langle A_{T_0}^\star A_R\boldsymbol{x}_R, \boldsymbol{x}_{T_0}\rangle| + |\langle (A_R^\star A_R - I)\,\boldsymbol{x}_R, \boldsymbol{x}_R\rangle| \\ &\quad\leq \delta_{T_0}\|\boldsymbol{x}_{T_0}\|_2^2 + \delta_r\|\boldsymbol{x}_R\|_2^2 + 2sr\varepsilon\|\boldsymbol{x}_{T_0}\|_2\|\boldsymbol{x}_R\|_2 \\ &\quad\leq (\max(\delta_{T_0}, \delta_r) + sr\varepsilon)(\|\boldsymbol{x}_{T_0}\|_2^2 + \|\boldsymbol{x}_R\|_2^2) \\ &\quad= \delta\|\boldsymbol{x}\|_2^2,\end{aligned}$$

where $\delta \equiv \max(\delta_{T_0}, \delta_r) + sr\varepsilon$. By (1b), we have $0 < \delta < 1$. Thus, A obeys the weak RIP with respect to T_0 of order r.

Example 2. Let T_0 and r be as in Example 1. Suppose the following hold:

(2a) $\max\{\|A_{T_0\cup R}\|;\ R \subset T_0^c \text{ and } |R| = r\} \leq 1$.

(2b) $\{A_{\{i\}};\ i \in T_0\}$ is linearly independent.

(2c) For any $R \subset T_0^c$ with $|R| = r$, $\{A_{\{j\}};\ j \in R\}$ is linearly independent.

It follows from (2a) and (2b) that A_{T_0} obeys the RIP of order s and from (2c) that $A_{T_0^c}$ obeys the RIP of order r. We denote by δ_{T_0} and δ_r the isometry constants of A_{T_0} and $A_{T_0^c}$, respectively. It is easily shown that $\delta_{T_0} = 1 - c_{T_0}^2$ and $\delta_r = 1 - \min_R c_R^2$. Furthermore, suppose the following (2d) and (2e) hold:

(2d) For $k \in \{1, 2, \ldots, n\}$, adding a vector $A_{\{k\}}'$ in $\boldsymbol{R}^{n-m}$ to the vector $A_{\{k\}}$ in $\boldsymbol{R}^m$, we construct a vector in $\boldsymbol{R}^n$ as follows:

$$B_{\{k\}} = \begin{pmatrix} A_{\{k\}} \\ A_{\{k\}}' \end{pmatrix}, \qquad \|B_{\{k\}}\|_2 = 1.$$

For any $R \subset T_0^c$ with $|R| = r$,

$$\{B_{\{i\}};\ i \in T_0\} \perp \{B_{\{j\}}; j \in R\}.$$

(2e) For any $i \in T_0$,

$$1 - \|A_{\{i\}}\|_2^2 \leq \left(\frac{1-\max(\delta_{T_0}, \delta_r)}{sr}\right)^2.$$

Then A obeys the weak RIP with respect to T of order r. Indeed, it follows from (2d) and (2e) that for any $i \in T_0$ and $j \in T_0^c$

$$\begin{aligned}\|A_{\{i\}}'\|_2^2 &= \|B_{\{i\}}\|_2^2 - \|A_{\{i\}}\|_2^2 = 1 - \|A_{\{i\}}\|_2^2 \\ &\leq \left(\frac{1-\max(\delta_{T_0}, \delta_r)}{sr}\right)^2,\end{aligned}$$

which implies by (2d) that

$$\begin{aligned}\langle A_{\{i\}}, A_{\{j\}}\rangle &= |\langle B_{\{i\}}, B_{\{j\}}\rangle - \langle A_{\{i\}}', A_{\{j\}}'\rangle| \\ &= |\langle A_{\{i\}}, A_{\{j\}}\rangle| \\ &\leq \|A_{\{i\}}'\|_2\|A_{\{j\}}'\|_2 \\ &\leq \|A_{\{i\}}'\|_2 \\ &\leq \frac{1-\max(\delta_{T_0}, \delta_r)}{sr}.\end{aligned}$$

Hence, (1a), (1b) and (1c) in Example 1 hold, and so A obeys the weak RIP with respect to T of order r.

Example 3. Let T_0 and r be as in Example 1. Suppose that the following hold:

(3a) $\max\{\|A_{T_o \cup R}\|;\ R \subset T_0^c \text{ and } |R| = r\} \leq 1$.

(3b) $\{A_{\{i\}};\ i \in T_0\}$ is an orthogonal system in $\boldsymbol{R}^m$.

(3c) For any $R \subset T_0^c$ with $|R| = r$, $\{A_{\{j\}};\ j \in R\}$ is linearly independent in $\boldsymbol{R}^m$ and orthogonal to $\{A_{\{i\}};\ i \in T_0\}$.

Then, A obeys the weak RIP with respect to T_0 of order r. Indeed, this follows from

$$\begin{aligned}\|A\boldsymbol{x}_{T_0 \cup R}\|_2^2 &= \|A\boldsymbol{x}_{T_0}\|_2^2 + \|A\boldsymbol{x}_R\|_2^2 + 2\,\langle A\boldsymbol{x}_{T_0}, A\boldsymbol{x}_R\rangle \\ &\geq (1 - \delta_{T_0})\,\|\boldsymbol{x}_{T_0}\|_2^2 + (1 - \delta_r)\,\|\boldsymbol{x}_R\|_2^2 \\ &\geq (1 - \max(\delta_{T_0}, \delta_r))\,\|\boldsymbol{x}\|_2^2.\end{aligned}$$

We remark that $\delta_{T_0} = 1 - \min\{\|A_{\{i\}}\|_2^2;\ i \in T_0\}$ and $\delta_r = 1 - \min_R c_R^2$.

The following example is a special case of Example 3 and it is useful itself.

Example 4. Let T_0 be a subset of $\{1, 2, \ldots, n\}$ with $|T_0| = s$ and R_0 be a subset of T_0^c with $|R_0| = r$. Suppose that an $m \times n$ matrix A satisfies the following conditions:

(4a) $\|A_{\{k\}}\|_2 \leq \frac{1}{\sqrt{s+r}}, \quad k = 1, 2, \ldots, n$.

(4b) $\{A_{\{i\}};\ i \in T_0 \cup R_0\}$ is an orthogonal system in $\boldsymbol{R}^m$.

(4c) $\{A_{\{j\}};\ j \in (T_0 \cup R_0)^c\}$ is contained in the linear span of $\{A_{\{i\}};\ i \in R_0\}$.

(4d) $\{A_{\{j\}};\ j \in R\}$ is linearly independent for each subset R of $(T_0 \cup R_0)^c$ with $|R| = r$.

Then A obeys the weak RIP with respect to T_0 of order r. Indeed, (3a) follows from (4a) and (3c) follows from (4b)–(4d). Hence, by Example 3 A obeys the weak RIP with respect to T_0 of order r.

4. Discussions

When analyzing data by using compressed sensing, it is common to use random matrices with no data structure. When matrices with data structure are used, it is very difficult to investigate whether these matrices obey RIP. Therefore, it seems useful to weaken the conditions of RIP. E. J. Candès and Y. Plan [3] have defined the notion of the weak RIP as a generalization of the RIP. In this paper, we have obtained almost the same results as for the case of the RIP. This is significant because it is much easier to construct matrices obeying the weak RIP than matrices obeying the RIP. In the case that we have some information about the data, that is, we know good locations T_0, it seems better to analyze data using the weak RIP. We believe the proposed definition has more potential applications to statistics and other fields than the original RIP.

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