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Scaling limit of d-inverse of Brownian motion with functional drift

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Abstract. The d-inverse is a generalized notion of inverse of a stochastic process having a certain tendency of increasing expectations. Scaling limit of the d-inverse of Brownian motion with functional drift is studied. Except for degenerate case, the class of possible scaling limits is proved to consist of the d-inverses of Brownian motion without drift, one with explosion in finite time, and one with power drift.

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1. Introduction

For (general) stock price $S = (S_t)_{t\geq 0}$, the European call option price with strike K and maturity t is given as

(1.1) $$C(t) := E\left[\max\left\{S_t - K, 0\right\}\right].$$

Suppose that the stock price is given as the *geometric Brownian motion* with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$:

(1.2) $$\mathrm{d}S_t = \sigma S_t \mathrm{d}B_t + \mu S_t \mathrm{d}t, \quad S_0 = s_0 \in (0,\infty),$$

where $B = (B_t)_{t\geq 0}$ denotes a one-dimensional standard Brownian motion. Letting $\widetilde{\mu} = \mu - \sigma^2/2$, we have an explicit expression of $S = S^{(\sigma,\mu)}$ as follows:

(1.3) $$S_t^{(\sigma,\mu)} = s_0 \exp\left(\sigma B_t + \widetilde{\mu} t\right).$$

If $\widetilde{\mu} = -\sigma^2/2$, then we may express $C(t)$ explicitly, in terms of the cumulative distribution function of the standard Gaussian

(1.4) $$\mathcal{N}(x) = \int_{-\infty}^{x} \mathrm{e}^{-x^2/2}\mathrm{d}x/\sqrt{2\pi},$$

as

(1.5) $$\begin{aligned} C(t) =& s_0 \mathcal{N}\left(-\frac{1}{\sigma\sqrt{t}}\log\frac{K}{s_0} + \frac{1}{2}\sigma\sqrt{t}\right) \\ &- K\mathcal{N}\left(-\frac{1}{\sigma\sqrt{t}}\log\frac{K}{s_0} - \frac{1}{2}\sigma\sqrt{t}\right), \end{aligned}$$

which is a special case of the well-known Black–Scholes formula. We may verify, by a direct computation, that $C(t)$ is increasing in $t > 0$; see Madan–Roynette–Yor [11].

Note that $S^{(\sigma,\mu)}$ is a submartingale if and only if $\mu \geq 0$. In this case, we can verify, without computing it explicitly, that $C(t)$ is increasing in $t > 0$. (In this paper, we mean non-decreasing by increasing.) More generally, for any increasing convex function φ, we may apply Jensen's inequality to see that, for any $0 < s < t$,

(1.6) $$E[\varphi(S_s)] \leq E[\varphi(E[S_t|\mathcal{F}_s])] \leq E[\varphi(S_t)].$$

In this sense, the submartingale property may be considered a tendency of increasing expectations.

To characterize another tendency of increasing expectations, The following notion was introduced by Madan–Roynette–Yor [10] and was developed by Profeta–Roynette–Yor [12]:

Definition 1.1. Let $R = (R_t)_{t\geq 0}$ denote a stochastic process taking values on $[0,\infty)$ defined on a measurable space equipped with a family of probability measures $(P_x)_{x\geq 0}$. Suppose that R is a.s. continuous and such that $P_x(R_0 = x) = 1$ for all $x \geq 0$.

(i) R is said *to admit an increasing pseudo-inverse* if $P_x(R_t \geq y)$ is increasing in $t \geq 0$ for all $y > x$ and if $P_x(R_t \geq y) \to 1$ as $t \to \infty$ for all $y > x$.

(ii) A family of random variables $(Y_{x,y})_{y>x}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called *pseudo-inverse* of R if for any $y > x$ it holds that

(1.7) $$P_x(R_t \geq y) = P(Y_{x,y} \leq t).$$

We would like here to introduce the following alternative notion, which is a slight modification of the pseudo-inverse:

Definition 1.2. Let $x_0 \in \mathbb{R}$. Let $X = (X_t)_{t\geq 0}$ be a stochastic process taking values in $[-\infty,\infty]$.

(i) X is called *d-increasing* on $[x_0,\infty)$ if $P(X_t \geq x)$ is increasing in $t \in (0,\infty)$ for all $x \in [x_0,\infty)$.

(ii) A family of random variables $(Y_x)_{x\geq x_0}$ is called *d-inverse* of X on $[x_0,\infty)$ if the following assertions

hold:
(iia) for any $x \in [x_0, \infty)$, the Y_x is a random variable taking values in $[0, \infty]$;
(iib) for any $x \in [x_0, \infty)$ and for a.e. $t \in (0, \infty)$, it holds that

$$P(X_t \geq x) = P(Y_x \leq t). \tag{1.8}$$

We note that X is d-increasing on $[x_0, \infty)$ if and only if X admits some d-inverse $(Y_x)_{x \geq x_0}$. We also note that if $P(X_t \geq x)$ is right-continuous in $t \in (0, \infty)$, then the identity (1.8) holds for all $t \in (0, \infty)$.

If $t \mapsto X_t$ is a.s. increasing, then X is d-increasing and its d-inverse is given by its inverse in the usual sense. The d-inverse may be a generalized notion of inverse in the sense of probability distribution.

Let S be a stochastic process such that $P(S_t \geq x)$ is right-continuous in $t \in (0, \infty)$. We note that S is d-increasing on $[x_0, \infty)$ if and only if $E[\varphi(S_t)]$ is increasing in $t > 0$ for all increasing (possibly non-convex) function φ whose support is contained in $[x_0, \infty)$ such that $E[\varphi(S_t)] < \infty$ for all $t > 0$. In fact, for the sufficiency, it holds that, for any $t > 0$,

$$E[\varphi(S_t)] = \varphi(x_0) P(S_t \geq x_0) + \int_{x_0}^{\infty} P(S_t \geq x) \mathrm{d}\varphi(x), \tag{1.9}$$

which shows that $E[\varphi(S_t)]$ is increasing in $t > 0$; the necessity is obvious since

$$E[1_{[x,\infty)}(S_t)] = P(S_t \geq x). \tag{1.10}$$

In particular, if S is a non-negative process such that $P(S_t \geq x)$ is right-continuous in $t \in (0, \infty)$, then the condition that S is d-increasing on $[0, \infty)$ is stronger than the one that S has the same one-dimensional marginals with a submartingale; see Remark 1.5.

In this paper, we confine ourselves to the class of processes of the form

$$B_t^{(\rho)} = B_t + \rho(t) \tag{1.11}$$

for some increasing function $\rho(t)$. We may call $B^{(\rho)}$ *Brownian motion with functional drift*. This process appears in *geometric Brownian motion with functional coefficients* as follows. Let $\sigma(t)$ and $\mu(t)$ be positive functions on $[0, \infty)$ and define

$$\mathrm{d}S_t = \sigma(t) S_t \mathrm{d}B_t + \mu(t) S_t \mathrm{d}t, \quad S_0 = s_0 > 0. \tag{1.12}$$

The resulting process $S = S^{(\sigma,\mu)}$ is given in the explicit form as

$$S_t^{(\sigma,\mu)} = s_0 \exp\left(\int_0^t \sigma(s) \mathrm{d}B_s + \int_0^t \widetilde{\mu}(s) \mathrm{d}s \right), \tag{1.13}$$

where $\widetilde{\mu}(t) = \mu(t) - \sigma^2(t)/2$. If we set $a(t) = \int_0^t \sigma(u)^2 \mathrm{d}u$, $b(t) = \int_0^t \widetilde{\mu}(u) \mathrm{d}u$ and set $\rho(t) = b(a^{-1}(t))$, then we obtain

$$S_{a^{-1}(t)}^{(\sigma,\mu)} = s_0 \exp\left(\beta_t + \rho(t)\right), \tag{1.14}$$

where $\beta = (\beta_t)_{t \geq 0}$ denotes a new Brownian motion.

The aim of this paper is to study scaling limit of the d-inverse on $[0, \infty)$ of $B^{(\rho)}$ for positive drift ρ. By *scaling limit* of d-inverse $Y^{(\rho)} = (Y_x^{(\rho)})_{x \geq 0}$ of $B^{(\rho)}$ we mean a process $Z = (Z_x)_{x \geq 0}$ such that

$$\frac{1}{\lambda} Y_{\phi_2(\lambda)x}^{(\phi_1(\lambda)\rho)} \xrightarrow[\lambda \to 0+]{\mathrm{d}} Z_x \quad \text{for all } x \in [0, \infty) \tag{1.15}$$

for some scaling functions ϕ_1 and ϕ_2. We assume that the ratio $\phi_2(\lambda)/\sqrt{\lambda}$ converges to a constant as $\lambda \to 0+$. We shall prove that the class of possible scaling limits consists, except for degenerate case, of the d-inverses of the following processes:

(i) Brownian motion without drift B_t;

(ii) *Brownian motion with explosion in finite time*: $B_t + \infty 1_{\{t \geq t_0\}}$, with $t_0 \in (0, \infty)$;

(iii) *Brownian motion with power drift*: $B_t + ct^{\alpha}$, with $c \in (0, \infty)$ and $\alpha \geq 1/2$.

Cases (i) and (ii) can be obtained from (iii) by taking limits; in fact, Case (i) can be obtained from (ii) as $t_0 \to \infty$ and Case (ii) can be obtained from (iii) by setting $c = t_0^{-\alpha}$ and letting $\alpha \to \infty$.

Here we make several remarks.

Remark 1.3. Monotonicity of more general option prices for more general stock processes have been studied by Hobson ([6], [7]), Henderson–Hobson ([3], [4]), and Kijima [9].

Remark 1.4. Let $X^{(1)}$ and $X^{(2)}$ be two random variables taking values in $[-\infty, \infty]$ and let $x_0 \in \mathbb{R}$. We write

$$X^{(1)} \leq_{\mathrm{st}} X^{(2)} \quad \text{on } [x_0, \infty) \tag{1.16}$$

if

$$P(X^{(1)} \geq x) \leq P(X^{(2)} \geq x) \quad \text{for all } x \in [x_0, \infty). \tag{1.17}$$

The relation $\leq_{\mathrm{st}}$ on $[x_0, \infty)$ is a partial order on the class of random variables. It may be called *usual stochastic order on* $[x_0, \infty)$ (see also Shaked–Shanthikumar [15]). We point out that a process $(X_t)_{t \geq 0}$ is d-increasing on $[x_0, \infty)$ if and only if $t \mapsto X_t$ is increasing in d-order on $[x_0, \infty)$.

Remark 1.5. Let $X^{(1)}$ and $X^{(2)}$ be two random variables taking values in $\mathbb{R}$. We write

$$X^{(1)} \leq_{\mathrm{icx}} X^{(2)} \tag{1.18}$$

if

$$E[\varphi(X^{(1)})] \leq E[\varphi(X^{(2)})] \quad \text{for all increasing convex function } \varphi. \tag{1.19}$$

The relation $\leq_{\mathrm{icx}}$ is a partial order on the class of random variables, so that it is called *increasing convex order* (see Shaked–Shanthikumar [15]). It is known (Kellerer [8]) that a process $(S_t)_{t \geq 0}$ is increasing in increasing convex order if and only if $(S_t)_{t \geq 0}$ has the same one-dimensional marginals with a submartingale. Interested readers are referred to Rothschild–Stiglitz ([13],[14]), Baker–Yor [1], and also Hirsch–Yor [5].

Remark 1.6. Profeta–Roynette–Yor [12] proved that a Bessel process admits pseudo-inverse if and only if the dimension is greater than one, and investigated several remarkable properties of its pseudo-inverse. See also Yen–Yor [16] for another related study of Bessel process.

This paper is organized as follows. In Section 2, we discuss d-inverses of several classes of processes and study scaling limit theorems of d-inverses. In Section 3, we study the inverse problem of scaling limits of d-inverses.

2. Discussions on d-increasing processes

For two random variables X and Y, we write $X \overset{\mathrm{d}}{=} Y$ if $P(X \le x) = P(Y \le x)$ for all $x \in \mathbb{R}$. For a family of random variables $(X^{(a)})_{a \in I}$ indexed by an interval I of $\mathbb{R}$, we write $X^{(a)} \xrightarrow{\mathrm{d}} X$ as $a \to b \in I$ for a random variable X if $P(X^{(a)} \le x) \to P(X \le x)$ as $a \to b$ for all $x \in \mathbb{R}$ such that $P(X = x) = 0$.

2.1. Transformations by increasing functions

For an increasing function $f : I \to [-\infty, \infty]$ defined on an subinterval I on $\mathbb{R}$, we denote its left-continuous inverse by $f^{-1} : \mathbb{R} \to [-\infty, \infty]$, i.e.:

$$f^{-1}(y) = \inf\{x \in I : f(x) \ge y\} \tag{2.1}$$

$$= \sup\{x \in I : f(x) < y\}, \tag{2.2}$$

where we adopt the usual convention that $\inf \emptyset = \sup I$ and $\sup \emptyset = \inf I$. By definition, we see that

$$f(x) \ge y \text{ implies } x \ge f^{-1}(y), \tag{2.3}$$

$$f(x) < y \text{ implies } x \le f^{-1}(y). \tag{2.4}$$

As a general remark, we give the following theorem.

Theorem 2.1. *Let $X = (X_t)_{t \ge 0}$ be a stochastic process such that $X_t \in [x_0, \infty)$ almost surely for all $t \ge 0$. Let $f : [x_0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to [0, \infty)$ be continuous increasing functions. Suppose that X admits a d-inverse $(Y_x)_{x \ge x_0}$. Then $\widehat{X} = (\widehat{X}_t)_{t \ge 0}$ defined by*

$$\widehat{X}_t = f\left(X_{g(t)}\right), \quad t \ge 0 \tag{2.5}$$

admits a d-inverse $\left(g^{-1}\left(Y_{f^{-1}(y)}\right)\right)_{y \ge f(x_0)}$.

Proof. Since f is continuous and increasing, we see that $f(f^{-1}(y)) = y$, and hence that $f(x) \ge y$ if and only if $x \ge f^{-1}(y)$. This proves that

$$P(f(X_{g(t)}) \ge y) = P(X_{g(t)} \ge f^{-1}(y)) \tag{2.6}$$

$$= P(Y_{f^{-1}(y)} \le g(t)) \tag{2.7}$$

$$= P\left(g^{-1}(Y_{f^{-1}(y)}) \le t\right). \tag{2.8}$$

The proof is complete. □

2.2. Brownian motion with functional drift

Theorem 2.2. *Let $\rho : [0, \infty) \to \mathbb{R}$ be a right-continuous function. Then the process $B_t^{(\rho)} = B_t + \rho(t)$ is d-increasing on $[0, \infty)$ if and only if the following condition is satisfied:*

$$\textbf{(A)} \quad \frac{\rho(t)}{\sqrt{t}} \text{ is increasing in } t > 0. \tag{2.9}$$

In this case, the d-inverse $(Y_x^{(\rho)})_{x \ge 0}$ is given by

$$Y_x^{(\rho)} \overset{\mathrm{d}}{=} \eta_x^{-1}(B_1) \quad \text{for all } x \ge 0, \tag{2.10}$$

where $\eta : (0, \infty) \to \mathbb{R}$ is the increasing function defined by

$$\eta_x(t) = \frac{\rho(t) - x}{\sqrt{t}}, \quad t > 0. \tag{2.11}$$

Proof. Since $B_t \overset{\mathrm{d}}{=} -\sqrt{t}B_1$, we have

$$P\left(B_t^{(\rho)} \ge x\right) = P\left(B_1 \le \eta_x(t)\right), \tag{2.12}$$

where η_x is defined as (2.11). Now $B^{(\rho)}$ is d-increasing if and only if $\eta_x(t)$ is increasing in $t > 0$ for all $x \ge 0$, which is equivalent to the condition **(A)**. □

In the remainder of this section, we discuss several particular classes of Browinan motion with functional drifts.

2.3. Brownian motion with explosion

Using $B_t \overset{\mathrm{d}}{=} \sqrt{t}B_1$, we obtain the following: The Brownian motion without drift, $B = B^{(0)}$, admits a d-inverse $Y^{(0)} = (Y_x^{(0)})_{x \ge 0}$. In fact, we have

$$Y_x^{(0)} \overset{\mathrm{d}}{=} \left(\frac{x}{B_1}\right)^2 1_{\{B_1 > 0\}} + \infty 1_{\{B_1 \le 0\}}, \quad x \ge 0. \tag{2.13}$$

For a constant $t_0 \in (0, \infty)$, the process $X = (X_t)_{t \ge 0}$ taking values in $(-\infty, \infty]$ defined by

$$X_t = B_t + \infty 1_{\{t \ge t_0\}}, \quad t \ge 0 \tag{2.14}$$

is called *Brownian motion with explosion in finite time*. It admits a d-inverse $Y = (Y_x)_{x \ge 0}$ given by

$$Y_x \overset{\mathrm{d}}{=} \min\left\{Y_x^{(0)}, t_0\right\}, \quad x \ge 0. \tag{2.15}$$

Theorem 2.3. *Let $\rho : [0, \infty) \to (0, \infty)$ be a right-continuous function satisfying the condition **(A)**. Let $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ be two functions. Suppose that there exist constants $t_0 \in (0, \infty]$ and $p \in [0, \infty)$ such that*

$$\textbf{(B)} \quad \begin{cases} \dfrac{\phi_1(\lambda)\rho(\lambda t)}{\sqrt{\lambda t}} \underset{\lambda \to 0+}{\longrightarrow} \begin{cases} 0 & \text{if } 0 < t < t_0, \\ \infty & \text{if } t > t_0, \end{cases} \\ \dfrac{\phi_2(\lambda)}{\sqrt{\lambda}} \underset{\lambda \to 0+}{\longrightarrow} p. \end{cases} \tag{2.16}$$

Then, for any $x \geq 0$, it holds that

$$\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \xrightarrow{\mathrm{d}} \min\left\{ Y^{(0)}_{px}, t_0 \right\} \quad \text{as } \lambda \to 0+. \tag{2.17}$$

In particular, for any $\lambda > 0$, it holds that

$$\frac{1}{\lambda} \min\left\{ Y^{(0)}_{\sqrt{\lambda}x}, t_0 \right\} \stackrel{\mathrm{d}}{=} \min\left\{ Y^{(0)}_{x}, t_0 \right\}. \tag{2.18}$$

Proof. Since $B_{\lambda t} \stackrel{\mathrm{d}}{=} \sqrt{\lambda} B_t$, we have

$$P\left(\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \leq t \right) \tag{2.19}$$

$$= P\left(B_{\lambda t} + \phi_1(\lambda)\rho(\lambda t) \geq \phi_2(\lambda) x \right) \tag{2.20}$$

$$= P\left(B_t + \frac{\phi_1(\lambda)\rho(\lambda t)}{\sqrt{\lambda}} \geq \frac{\phi_2(\lambda)}{\sqrt{\lambda}} x \right). \tag{2.21}$$

The last quantity converges as $\lambda \to 0+$ to $P(B_t \geq px)$ if $t < t_0$ and to 1 if $t > t_0$. Since we have

$$P\left(\min\left\{ Y^{(0)}_{px}, t_0 \right\} \leq t \right) = \begin{cases} P(B_t \geq px) & \text{if } t < t_0, \\ 1 & \text{if } t \geq t_0, \end{cases} \tag{2.22}$$

we obtain (2.17). The scale invariance property (2.18) is obvious. The proof is now complete. □

2.4. Brownian motion with constant drift

By Theorem 2.2, we see that the Brownian motion with constant drift $B^{(c\cdot)} = (B_t + ct)_{t \geq 0}$ admits a d-inverse $Y^{(c\cdot)} = (Y^{(c\cdot)}_x)_{x \geq 0}$ if and only if $c \in [0, \infty)$. If $c \in (0, \infty)$, i.e., except for the Brownian case, we obtain, for $x \geq 0$,

$$Y^{(c\cdot)}_x \stackrel{\mathrm{d}}{=} \left(\frac{B_1 + \sqrt{B_1^2 + 4cx}}{2c} \right)^2. \tag{2.23}$$

We remark that, for any $x \geq 0$,

$$Y^{(c\cdot)}_x \xrightarrow{\mathrm{d}} Y^{(0)}_x \quad \text{as } c \to 0+. \tag{2.24}$$

We also remark the following: Using $B_t \stackrel{\mathrm{d}}{=} -tB_{1/t}$, we can easily see that

$$Y^{(c\cdot)}_x \stackrel{\mathrm{d}}{=} \frac{1}{Y^{(x\cdot)}_c} \quad \text{for all } c \geq 0 \text{ and } x \geq 0. \tag{2.25}$$

Scaling property of Brownian motion with constant drifts will be discussed in the next section in a more general setting.

The geometric Brownian motion $S = S^{(\sigma,\mu)}$ with constant volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$ given as (1.3) may be represented as $S^{(\sigma,\mu)}_t = f(B^{((\widetilde{\mu}/\sigma)t)}_t)$ where $f(x) = s_0 \exp(\sigma x)$. Hence we may apply Theorem 2.1 and obtain the following: $S^{(\sigma,\mu)}$ admits a d-inverse $(T^{(\sigma,\mu)}_s)_{s \geq s_0}$ if and only if $\widetilde{\mu} = \mu - \sigma^2/2 \geq 0$. In this case, we have

$$T^{(\sigma,\mu)}_s \stackrel{\mathrm{d}}{=} Y^{((\widetilde{\mu}/\sigma)\cdot)}_{f^{-1}(s)} \quad \text{for all } s \geq s_0. \tag{2.26}$$

2.5. Brownian motion with power drift

For $\alpha \in [0, \infty)$ and $c \in [0, \infty)$, we define

$$R^{(c,\alpha)}_t = B_t + ct^\alpha, \quad t \geq 0 \tag{2.27}$$

and we call $R^{(c,\alpha)} = (R^{(c,\alpha)}_t)_{t \geq 0}$ a *Brownian motion with power drift.* By Theorem 2.2, we see that $R^{(c,\alpha)}$ admits a d-inverse $(Z^{(c,\alpha)}_x)_{x \geq 0}$ if and only if $\alpha \geq 1/2$.

The following theorem tells us that the class of the d-inverses of Brownian motion with power drifts appear as scaling limits, and consequently, satisfy scale invariance property.

Theorem 2.4. *Let $\rho : [0, \infty) \to (0, \infty)$ be a right-continuous function satisfying the condition* **(A)**. *Let $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ be two functions. Suppose there exist $\alpha \geq 1/2$, $c \in (0, \infty)$ and $p \in [0, \infty)$ such that*

$$\textbf{(RV)} \quad \begin{cases} \dfrac{\rho(\lambda t)}{\rho(\lambda)} \underset{\lambda \to 0+}{\longrightarrow} t^\alpha, \\ \dfrac{\rho(\lambda)}{\sqrt{\lambda}} \phi_1(\lambda) \underset{\lambda \to 0+}{\longrightarrow} c, \\ \dfrac{1}{\sqrt{\lambda}} \phi_2(\lambda) \underset{\lambda \to 0+}{\longrightarrow} p. \end{cases} \tag{2.28}$$

Then, for any $x \geq 0$, it holds that

$$\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \xrightarrow{\mathrm{d}} Z^{(c,\alpha)}_{px} \quad \text{as } \lambda \to 0+. \tag{2.29}$$

In particular, for any $\lambda > 0$, it holds that

$$\frac{1}{\lambda} Z^{\left(c\lambda^{(1/2)-\alpha}, \alpha\right)}_{\sqrt{\lambda}x} \stackrel{\mathrm{d}}{=} Z^{(c,\alpha)}_x. \tag{2.30}$$

Remark 2.5. The condition **(RV)** asserts that the functions ρ, ϕ_1 and ϕ_2 (if $p \in (0, \infty)$) are regularly varying at $0+$ of index α, $(1/2) - \alpha$, and $1/2$, respectively.

Proof of Theorem 2.4. Since $B_{\lambda t} \stackrel{\mathrm{d}}{=} \sqrt{\lambda} B_t$, we have

$$P\left(\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \leq t \right) \tag{2.31}$$

$$= P\left(B_t + \frac{\rho(\lambda)}{\sqrt{\lambda}} \phi_1(\lambda) \cdot \frac{\rho(\lambda t)}{\rho(\lambda)} \geq \frac{\phi_2(\lambda)}{\sqrt{\lambda}} x \right) \tag{2.32}$$

$$\underset{\lambda \to 0+}{\longrightarrow} P(B_t + ct^\alpha \geq px) \tag{2.33}$$

$$= P(Z^{(c,\alpha)}_{px} \leq t). \tag{2.34}$$

Now we have obtained (2.29). The scale invariance property (2.30) is obvious. The proof is complete. □

3. Scaling limits for the class of d-inverses

In what follows, by *measurable* we mean Lebesgue measurable.

Theorem 3.1. *Let $\rho : [0,\infty) \to [0,\infty)$ be a right-continuous function satisfying the condition* **(A)**. *Suppose that, for some measurable functions $\phi_1, \phi_2 : (0,\infty) \to (0,\infty)$ and for some family $Z = (Z_x)_{x \ge 0}$ of $[0,\infty]$-valued random variables, it holds that*

$$\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \xrightarrow[\lambda \to 0+]{\mathrm{d}} Z_x \quad \text{for all } x \ge 0. \tag{3.1}$$

Suppose, moreover, that there exists a constant $p \in [0,\infty)$ such that

$$\frac{\phi_2(\lambda)}{\sqrt{\lambda}} \xrightarrow[\lambda \to 0+]{} p. \tag{3.2}$$

Then either one of the following four assertions holds:

(i) $\phi_1(\lambda)\rho(\lambda t)/\sqrt{\lambda} \xrightarrow[\lambda \to 0+]{} 0$ for all $t > 0$. In this case,

$$Z_x \overset{\mathrm{d}}{=} Y^{(0)}_{px} \quad \text{for all } x \ge 0. \tag{3.3}$$

(ii) The condition **(B)** *holds for some $t_0 \in (0,\infty)$. In this case,*

$$Z_x \overset{\mathrm{d}}{=} \min\left\{ Y^{(0)}_{px}, t_0 \right\} \quad \text{for all } x \ge 0. \tag{3.4}$$

(iii) The condition **(RV)** *holds for some $\alpha \ge 1/2$ and $c \in (0,\infty)$. In this case,*

$$Z_x \overset{\mathrm{d}}{=} Z^{(c,\alpha)}_{px} \quad \text{for all } x \ge 0. \tag{3.5}$$

(iv) (Degenerate case.) $P(Z_x = 0) = 1$ for all $x \in (0,\infty)$.

Proof. Let $x \ge 0$. Denote $F_x(t) = P(Z_x \le t)$ for $t \ge 0$ and denote by $C(F_x)$ the set of continuity point of F_x. We note that

$$P\left(\frac{1}{\lambda} Y^{(\phi_1(\lambda)\rho)}_{\phi_2(\lambda)x} \le t \right) \tag{3.6}$$

$$= P\left(B_{\lambda t} + \phi_1(\lambda)\rho(\lambda t) \ge \phi_2(\lambda)x \right) \tag{3.7}$$

$$= P\left(B_1 + \phi_1(\lambda)\frac{\rho(\lambda t)}{\sqrt{\lambda t}} \ge \frac{\phi_2(\lambda)}{\sqrt{\lambda t}}x \right). \tag{3.8}$$

By the assumption (3.1), we see that

$$P\left(B_1 + \phi_1(\lambda)\frac{\rho(\lambda t)}{\sqrt{\lambda t}} - \frac{\phi_2(\lambda)}{\sqrt{\lambda t}}x \in [0,\infty) \right) \xrightarrow[\lambda \to 0+]{} P(Z_x \le t) \quad \text{for all } t \in C(F_x) \cap (0,\infty). \tag{3.9}$$

Hence there exists a function $g_x : C(F_x) \cap (0,\infty) \to [-\infty,\infty]$ such that

$$\phi_1(\lambda)\frac{\rho(\lambda t)}{\sqrt{\lambda t}} - \frac{\phi_2(\lambda)}{\sqrt{\lambda t}}x \xrightarrow[\lambda \to 0+]{} g_x(t) \quad \text{for all } t \in C(F_x) \cap (0,\infty). \tag{3.10}$$

Since ρ satisfies the condition **(A)** and since $C(F_x)$ is dense in $\mathbb{R}$, we see that g_x is increasing, and hence we may extend g_x on $[0,\infty)$ so that it is right-continuous. Now we obtain, for any $x \ge 0$,

$$Z_x \overset{\mathrm{d}}{=} g_x^{-1}(B_1). \tag{3.11}$$

Let us write g simply for g_0. Noting that g is an increasing function taking values in $[0,\infty]$. we divide into the following four distinct cases.

(i) *The case where $g(t) = 0$ for all $t > 0$.*

Let $x \ge 0$ be fixed. By the assumption (3.2) and by (3.10), we obtain

$$g_x(t) = -px/\sqrt{t}, \quad t > 0. \tag{3.12}$$

From this and (3.11), we obtain

$$P(Z_x \le t) = P(Y^{(0)}_{px} \le t), \quad t > 0. \tag{3.13}$$

This proves (3.3). The proof of Claim (i) is now complete.

(ii) *The case where there exist a point $t_0 \in (0,\infty)$ such that*

$$g(t) \begin{cases} = 0 & \text{if } 0 < t < t_0, \\ = \infty & \text{if } t > t_0. \end{cases} \tag{3.14}$$

Let $x \ge 0$. By the assumption (3.2) and by (3.10), we obtain

$$g_x(t) = \begin{cases} -px/\sqrt{t} & \text{if } 0 < t < t_0, \\ \infty & \text{if } t > t_0. \end{cases} \tag{3.15}$$

From this and (3.11), we obtain

$$P(Z_x \le t) = \begin{cases} P(Y^{(0)}_{px} \le t) & \text{if } 0 \le t < t_0, \\ 1 & \text{if } t \ge t_0. \end{cases} \tag{3.16}$$

This proves (3.4). The proof of Claim (ii) is now complete.

(iii) *The case where there are two points $t_0, t_1 \in C(F_0) \cap (0,\infty)$ with $t_0 < t_1$ such that $0 < g(t_0) \le g(t_1) < \infty$.*

Since g is increasing, we see that

$$0 < g(t) < \infty \quad \text{for all } t \in C(F_0) \cap [t_0, t_1]. \tag{3.17}$$

By (3.10), we have, for any $t \in C(F_0) \cap [t_0, t_1]$,

$$\frac{\rho(\lambda t)}{\rho(\lambda t_0)} = \frac{\phi_1(\lambda)\frac{\rho(\lambda t)}{\sqrt{\lambda t}}}{\phi_1(\lambda)\frac{\rho(\lambda t_0)}{\sqrt{\lambda t_0}}} \cdot \frac{\sqrt{t}}{\sqrt{t_0}} \xrightarrow[\lambda \to 0+]{} \frac{g(t)}{g(t_0)} \cdot \frac{\sqrt{t}}{\sqrt{t_0}} \in (0,\infty). \tag{3.18}$$

Since $C(F_0) \cap [t_0, t_1]$ has positive Lebesgue measure, we may apply Characterisation Theorem ([2, Theorem 1.4.1]) to see that the convergence (3.18) and consequently (3.10) are still valid for all $t \in (0,\infty)$, and that

$$\frac{g(t)}{g(t_0)} \cdot \frac{\sqrt{t}}{\sqrt{t_0}} = t^{\alpha}, \quad t \in (0,\infty) \tag{3.19}$$

for some $\alpha \in \mathbb{R}$. Since g is increasing, we have $\alpha \ge 1/2$. We obtain

$$g(t) = ct^{\alpha - 1/2}, \quad t \in (0,\infty) \tag{3.20}$$

for some $c \in (0, \infty)$. Hence, by (3.18) and (3.10), we obtain

$$\rho(\lambda t) / \rho(\lambda) \underset{\lambda \to 0+}{\longrightarrow} t^\alpha \quad \text{and} \quad \frac{\rho(\lambda)}{\sqrt{\lambda}} \phi_1(\lambda) \underset{\lambda \to 0+}{\longrightarrow} c. \tag{3.21}$$

Now we have seen that the condition **(RV)** is satisfied. The proof of Claim (iii) is now completed by Theorem 2.4.

(iv) *The case where $g(t) = \infty$ for all $t > 0$.*

In this case, by the assumption (3.2) and by (3.10), we obtain $g_x(t) = \infty$ for all $t > 0$ and $x \geq 0$. By (3.11), we obtain $P(Z_x = 0) = 1$ for all $x \geq 0$. The proof of Claim (iv) is now complete. □

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