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Algebraic instability caused by acoustic modes in supersonic shear flows

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Abstract. Perturbations in a shear flow exhibit rather complex behavior – waves may grow algebraically even when the spectrum of disturbances is entirely neutral (no exponential instability). A shear flow brings about non-selfadjoint property, invalidating the standard notion of dispersion relations, and it also produces a continuous spectrum that is a characteristic entity in an infinite-dimension phase space. This paper solves an initial value problem using the Laplace transform and presents a new-type of algebraic instability that is caused by resonant interaction between acoustic modes (point spectrum) and vortical continuum mode (continuous spectrum). Such a resonance is possible when variation of velocity shear is comparable to sound speed.

Keywords. compressible fluid, shear flow, spectrum of non-selfadjoint operator, algebraic instability,

1. Introduction

Waves in shear flows exhibit a variety of complex behavior that may not be decomposed into oscillating eigenmodes. The difficulty arises from the non-orthogonality of the eigenmodes of the non-selfadjoint (non-normal) generator. Moreover, in inviscid shear flows, there exists continuous spectrum as well as point (or discrete) spectrum, and the degeneracy of some point spectrum into the continuous spectrum requires careful mathematical consideration.

In this paper, we present a new-type of instability in inviscid compressible shear flow, which glows algebraically in time through a resonant interaction between the acoustic mode (a point spectrum) and the continuum of the vortical modes (a continuous spectrum). We note that such a resonant interaction of different modes is possible because they are not *orthogonal* by the non-selfadjointness. And for a resonance to occur, the phase speed of the acoustic wave must belong to the range of the shear-flow velocity, which requires that the shear flow must have supersonic region.

The study of instabilities in shear flow has a long history, and is still incomplete to understand, for example, the onset of turbulence [1, 2]. While the normal mode analysis (or the eigenvalue problem) has found a number of hydrodynamic instabilities (such as the Kelvin-Helmholtz instability, Rayleigh-Taylor instability and so on), it falls short to explain some experiments. For example, in the plane Couette flow, instabilities are experimentally observed for large Reynolds numbers $Re \gtrsim 350$, whereas the normal mode analyses predict stability for all Re.

The nonmodal approach has been gaining importance recently, while it was already present since Kelvin [3]. Algebraic instability of inviscid parallel shear flows was investigated by Ellingsen and Palm [4] and Landahl [5] in the incompressible case, and by Hanifi and Henningson [6] in the compressible case. These studies showed that three-dimensional disturbances which are uniform in the streamwise direction grows linearly with time. The physical explanation given by Landahl [5] (see also Ref. [2]) is so-called the *lift-up* effect; since the normal velocity fluctuation advects the fluid elements across the mean shear layer, the streamwise velocity fluctuation increases such that the momentum is conserved.

We can attribute this transient growth mathematically to the non-selfadjoint property of the linearized fluid equation. The recent our studies [7, 8] are devoted to understanding various inviscid algebraic instabilities in terms of degenerate spectra of non-selfadjoint operator. If we write the equations for disturbances in an evolutionary form like $i\partial_t\varphi = \mathcal{L}\varphi$ (φ: disturbance), the operator $\mathcal{L}$ is generally non-selfadjoint and, moreover, it is sometimes similar to Jordan's block as was pointed out by Arnold [9]. Recall that the spectral theory for matrices is established by the Jordan canonical form, where multiply-degenerated eigenvalues of a non-selfadjoint matrix may have nilpotent parts called Jordan's blocks (see, for example, Ref. [10]). If the operator $\mathcal{L}$ has Jordan's blocks, the resonance between eigenmodes leads to algebraic growth of their amplitudes. However, the fluid system has an infinite degree of freedom, and the disturbances in inviscid shear flow come with difficulty of the continuous spectrum. While the spectral theory for non-selfadjoint operators is still under development in mathematics, its progress serves to the accurate prediction of various hydrodynamic instabilities and will be fruitful for an industrial purpose.

In this work, a considerable attention is paid to the existence of the continuous spectrum in disturbances. The asymptotic behavior of the continuum mode was studied by Eliassen *et al.* [11] and Case [12] for incompressible shear flows. While the compressibility does not change their results basically, it will be shown that some point spectra of the acoustic modes can enter the range of the continuous spectrum when the shear flow is supersonic. They constitute another Jordan's block, and the lift-up effect by the acoustic mode elicits the linear growth of the continuum modes locally at the resonant point (which coincides with the so-called critical layer).

In the next section, we will present the evolution equation for three-dimensional disturbances in compressible shear flow. Some preliminary consideration will be made in reference to the earlier works.

In Sec. 3, the spectrum of the evolution equation will be investigated by the Laplace transform. Our analysis reproduces the result of the normal mode analysis by Lees and Lin [13] and Mack [14]. The existence of the continuous spectrum, as well as the discrete acoustic modes, and the possibility of their degeneracies will be highlighted. The asymptotic behavior will be estimated in Sec. 4 by means of the inverse Laplace transform. The algebraic instability of the continuum modes occurs due to the resonance at the degeneracy.

The degeneracy of point and continuous spectra is possible when the resonant point of the acoustic mode is a *generalized* inflection point of the shear flow [13] (which will be confirmed also in Sec. 3). The linear shear flow is therefore a simplest example that seems to be algebraically unstable. The stability of the plane Couette flow will be considered in Sec. 5, where the algebraic growth of disturbances indeed shows up due to the resonance between acoustic modes and continuum mode.

2. Spectrum of disturbances in bounded shear flow

The inviscid compressible fluids are governed by

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{1}$$

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \tag{2}$$

where ρ and $\mathbf{v}$ respectively denote mass density and flow velocity. The pressure p is a given function of only ρ by assuming the isentropic fluid. We consider the linear stability of a shear flow $\mathbf{v} = (0, v_y(x), v_z(x))$ with an uniform density $\rho \equiv 1$. Let the domain be infinite in the y and z directions and bounded in the x direction by two walls at $x = x_0$ and $x = x_1$. To simplify the analysis, the shear profile is assumed to be smooth and strictly increasing function;

$$v_y'(x) > 0 \quad \text{and} \quad v_z'(x) > 0 \quad \text{for all } x \in [x_0, x_1]. \tag{3}$$

In this paper, the prime ($'$) always denotes the derivative of the basic flow with respect to x. The linearization about this basic flow gives an evolution equation for perturbations $\tilde{\mathbf{v}}$ and $\tilde{p}$. Due to the uniformity in the y and z directions, it is sufficient to consider a single Fourier component $\tilde{\mathbf{v}}, \tilde{p} \propto e^{ik_y y + ik_z z}$, where the wavenumbers k_y and k_z are arbitrary real numbers. Moreover, we perform the change of

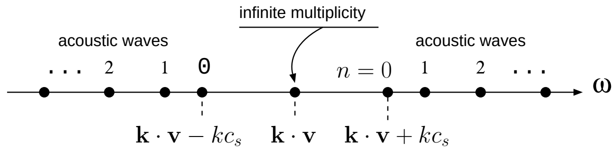

Figure 1: Spectrum of disturbances without shear flow

variables, $\tilde{v} = \tilde{v}_x$, $\tilde{u} = ik_y\tilde{v}_y + ik_z\tilde{v}_z$ and $\tilde{w} = ik_y\tilde{v}_z - ik_z\tilde{v}_y$, which is so-called Squire's transformation [15]. The linearized fluid equations for functions $\tilde{w}, \tilde{v}, \tilde{u}, \tilde{p}$ of (x, t) are written in an evolutionary form

$$i\partial_t \begin{pmatrix} \tilde{w} \\ \tilde{v} \\ \tilde{u} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \mathbf{k}\cdot\mathbf{v} & (\mathbf{k}\times\mathbf{v}')_x & 0 & 0 \\ 0 & \mathbf{k}\cdot\mathbf{v} & 0 & -i\partial_x \\ 0 & \mathbf{k}\cdot\mathbf{v}' & \mathbf{k}\cdot\mathbf{v} & ik^2 \\ 0 & -ic_s^2\partial_x & -ic_s^2 & \mathbf{k}\cdot\mathbf{v} \end{pmatrix} \begin{pmatrix} \tilde{w} \\ \tilde{v} \\ \tilde{u} \\ \tilde{p} \end{pmatrix}, \tag{4}$$

where we have used the wavenumber vector $\mathbf{k} = (0, k_y, k_z)$ in the y-z plane, the sound speed $c_s^2 = \partial p/\partial\rho|_{\rho=1} = \text{const.}$, $k = (k_y^2 + k_z^2)^{1/2}$ and $(\mathbf{k}\times\mathbf{v}')_x = k_y v_z'(x) - k_z v_y'(x)$. The boundary condition is given by

$$\tilde{v} = 0 \quad \text{and} \quad \partial_x\tilde{p} = 0 \quad \text{at } x = x_0, x_1. \tag{5}$$

When the shear is absent everywhere $\mathbf{v}'(x) \equiv 0$, the generator of the system is clearly selfadjoint. It is straightforward to show that infinite number of discrete eigenvalues are determined by the dispersion relation $(\omega - \mathbf{k}\cdot\mathbf{v})[(\omega - \mathbf{k}\cdot\mathbf{v})^2 - (n^2\pi^2/(x_1 - x_0)^2 + k^2)c_s^2] = 0$ where $n(= 0, 1, 2, \dots)$ is the quantum number labeling the structure in the x direction. The acoustic modes appear as an infinite number of real point spectra $\sigma_p = \{\omega_n\} \subset \mathbb{R}$ as illustrated in Figure 1, where $\{\omega_n\}$'s are distributed in the intervals $[-\infty, \mathbf{k}\cdot\mathbf{v} - kc_s]$ and $[\mathbf{k}\cdot\mathbf{v} + kc_s, \infty]$. There is also an infinite multiplicity at $\omega = \mathbf{k}\cdot\mathbf{v}$.

In the presence of velocity shear, the generator becomes non-selfadjoint and, moreover, the infinite multiplicity converts into the continuous spectrum. From (4), we can see that the interaction among $\tilde{v}$, $\tilde{u}$ and $\tilde{p}$ makes a closed system, whereas the normal component of the vorticity fluctuation $\tilde{w}$ is forced by $\tilde{v}$ alone. We will refer to the closed system $(\tilde{v}, \tilde{u}, \tilde{p})$ as Rayleigh part of (4), since it is essentially equivalent to the problem of two-dimensional disturbances, namely, the compressible version of the Rayleigh equation [12, 13]. As will be shown in the next section, this Rayleigh part possesses a real continuous spectrum

$$\sigma_c := \{\mathbf{k}\cdot\mathbf{v}(x) \in \mathbb{R};\ x \in [x_0, x_1]\}, \tag{6}$$

and also point spectra corresponding to the acoustic modes and the vortical (Kelvin-Helmholtz) mode in the complex frequency plane. The latter global eigenmodes may be exponentially unstable, $\mathrm{Im}(\omega_n) > 0$, as shown by the normal mode analysis [14, 16].

The equation for $\tilde{w}$, called the (inviscid) Squire equation, has another continuous spectrum σ_c that is exactly overlapping with the Rayleigh's one. Since this continuum mode represents the stretching of the normal component of vorticity and its frequency is always neutral ($\sigma_c \subset \mathbb{R}$), Squire claimed that it is sufficient to study only the stability of two-dimensional disturbances, i.e., the Rayleigh part. This Squire's theorem is correct within the framework of the normal mode analysis.

However, Arnold [9] pointed out the resemblance of the formal structure of the Rayleigh-Squire equation to the Jordan's block. He inferred the possibility of the linear increase of vortex fluctuation $\tilde{w}$ with time even in the absence of exponential instability. For inviscid incompressible shear flows, this algebraic instability was confirmed by Ellingsen and Palm [4], who considered an unidirectional shear flow, say $\mathbf{v} = (0, v_y(x), 0)$, and a special class of disturbances that are uniform in the streamwise direction, $k_y = 0$, $k_z \neq 0$. Since $\mathbf{k} \cdot \mathbf{v} \equiv 0$ and $(\mathbf{k} \times \mathbf{v})_x \neq 0$ in this special case, it is easily verified that a purely linear growth of $\tilde{w}$ occurs at the zero eigenfrequency $\sigma_c = \{0\}$ (the *lift-up* effect [5]).

The difficulty of the continuous spectrum arises when general disturbances are taken into account. The result of the initial value problem, given by Eliassen *et al.* [11] and Case [12], indicates that the normal component of the velocity fluctuation $\tilde{v}$ damps at least as fast as t^{-2} for $t \to \infty$ due to the phase mixing effect of the continuous spectrum. This short-lived $\tilde{v}$ cannot fully elicit the linear growth of the Squire part. Landahl [5] considered a localized initial disturbance and showed that its energy grows linearly with time, where the disturbed region increases linearly in time while the amplitude of $\tilde{w}$ saturates.

Now, let us turn to the compressible case (4). Hanifi and Henningson [6] showed that the algebraically unstable solution still exists for the same incompressible disturbances as Ref. [4]. In this work, we propose another possibility of algebraic instability that occurs in general disturbances. It will be shown that the point spectra of the acoustic modes may overlap the continuous spectrum when the variation of the shear flow exceeds the sound speed c_s (the width of σ_c reaches the fundamental mode $\omega_0 = kc_s$). The acoustic mode is, then, expected to drive new algebraic instability of the continuum modes in the Squire part $\tilde{w}$ unless $(\mathbf{k} \times \mathbf{v}')_x = 0$.

A naive picture of wave propagation may deny the existence of such an eigenmode, because an eigenmode, representing a standing wave created by the interference of acoustic waves reflected by the walls, must persist against the supersonic flow. Indeed, the eikonal analysis (see Appendix A) cannot explain the creation of such standing waves. However, if the corresponding critical point occurs at a *generalized* inflection point of the shear flow (as discussed by Lees and Lin [13]), the tunneling effect allows propagation of acoustic waves across the supersonic flow, and creates eigenmodes in the range of the continuous spectrum. In particular, such eigenmodes are abundant in linear shear flow, that is, the plane Couette flow. This fact will be confirmed in the next section.

3. Existence of neutral acoustic modes in continuous spectrum

The stability of the inviscid compressible shear flow was studied by Lees and Lin [13], Mack [14] and the subsequent works (see Ref. [16]). They investigated exponential instabilities of the acoustic modes by solving the eigenvalue problem of (4). In this section, we will take the nonmodal approach by exploiting the Laplace transform analysis in order to deal with the continuous spectrum and possible algebraic (or non-exponential) behavior.

The Laplace transform of $\tilde{p}(x,t)$ is given by

$$P(x,\Omega) = \int_0^\infty \tilde{p}(x,t) e^{i\Omega t} dt, \tag{7}$$

for $\Omega \in \mathbb{C}$ with sufficiently large $\mathrm{Im}(\Omega)$. Similarly, we transform $\tilde{v}(x,t), \tilde{u}(x,t), \tilde{w}(x,t)$ into $V(x,\Omega), U(x,\Omega), W(x,\Omega)$. The evolution equation (4) is then transformed into

$$(\Omega - \mathbf{k}\cdot\mathbf{v})P + c_s^2 i(\partial_x V + U) = i\, \tilde{p}|_{t=0}\,, \tag{8}$$
$$(\Omega - \mathbf{k}\cdot\mathbf{v})V + i\partial_x P = i\, \tilde{v}|_{t=0}\,, \tag{9}$$
$$(\Omega - \mathbf{k}\cdot\mathbf{v})U - \mathbf{k}\cdot\mathbf{v}'V - ik^2 P = i\, \tilde{u}|_{t=0}\,, \tag{10}$$
$$(\Omega - \mathbf{k}\cdot\mathbf{v})W - (\mathbf{k}\times\mathbf{v}')_x V = i\, \tilde{w}|_{t=0}\,, \tag{11}$$

where the right hand sides represent the initial values, $\tilde{p}|_{t=0} = \tilde{p}(x,0)$.

The Rayleigh part, (8) - (10), should be solved simultaneously, and afterward W will be obtained from (11). By eliminating V and U from (8) - (10), we get

$$\partial_x^2 P + 2\frac{M'}{\hat{\Omega} - M}\partial_x P + k^2[(\hat{\Omega} - M)^2 - 1]P \tag{12}$$
$$= i\frac{k}{c_s}(\hat{\Omega} - M)\, \tilde{p}|_{t=0} + (\partial_x \tilde{v} + \tilde{u})|_{t=0} + 2\frac{M'}{\hat{\Omega} - M}\, \tilde{v}|_{t=0}\,,$$

where $\hat{\Omega} = \Omega/kc_s$ and $M(x) = \mathbf{k}\cdot\mathbf{v}(x)/kc_s$ respectively represent the Mach numbers of the phase speed and the basic flow (in the direction parallel to $\mathbf{k}$). This inhomogeneous ordinary differential equation for $P(x,\Omega)$ is solved under the Neumann boundary condition at $x = x_0, x_1$.

For fixed $\Omega \in \mathbb{C}$, the solution of (12) is generally represented by

$$P(x,\Omega) = C^\dagger(\Omega)P^\dagger(x,\Omega) + C^\ddagger(\Omega)P^\ddagger(x,\Omega) + P^\sharp(x,\Omega), \tag{13}$$

where $P^\dagger(x,\Omega)$ and $P^\ddagger(x,\Omega)$ denote the two linearly independent homogeneous solutions and the last term $P^\sharp(x,\Omega)$ denotes a particular solution to the inhomogeneous terms in the right hand side of (12). The coefficients $C^\dagger(\Omega)$ and $C^\ddagger(\Omega)$ must be determined such that the boundary condition $\partial_x P(x_0,\Omega) = \partial_x P(x_1,\Omega) = 0$ holds, which results in

$$\begin{pmatrix} C^\dagger(\Omega) \\ C^\ddagger(\Omega) \end{pmatrix} = \frac{1}{D(\Omega)} \times \begin{pmatrix} -\partial_x P^\ddagger(x_1,\Omega) & \partial_x P^\ddagger(x_0,\Omega) \\ \partial_x P^\dagger(x_1,\Omega) & -\partial_x P^\dagger(x_0,\Omega) \end{pmatrix} \begin{pmatrix} \partial_x P^\sharp(x_0,\Omega) \\ \partial_x P^\sharp(x_1,\Omega) \end{pmatrix}, \tag{14}$$

with

$$D(\Omega) := \partial_x P^\dagger(x_0,\Omega)\partial_x P^\ddagger(x_1,\Omega) - \partial_x P^\ddagger(x_0,\Omega)\partial_x P^\dagger(x_1,\Omega). \tag{15}$$

The set of Ω at which $P(x,\Omega)$ is not regular is the spectrum. If $D(\Omega)$ becomes zero at, say, $\Omega = \omega_0 \in \mathbb{C}$, $P(x,\Omega)$ has an isolated pole at $\Omega = \omega_0$. Such ω_0 is classified into point spectra, and hence $D(\Omega) = 0$ plays the role of the dispersion relation. The acoustic modes and the Kelvin-Helmholtz modes belong to this class.

The Rayleigh part $(\tilde{v}, \tilde{u}, \tilde{p})$ has the continuous spectrum σ_c, defined in (6), associated with the singularity $1/(\hat{\Omega}-M)$ in (12). For each $\omega \in \sigma_c$, there exists a resonant point $x_c(\omega) \in [x_0, x_1]$ that satisfies $\omega - \mathbf{k}\cdot\mathbf{v}(x_c) = 0$. Indeed, the solution $P(x,\Omega)$ has a logarithmic singularity at $x = x_c(\omega)$ when $\Omega \to \omega \in \sigma_c$.

The well-known Frobenius' method [17] constructs the homogeneous solutions in series near the singularity. Let us define the reciprocal function of $\Omega = \mathbf{k}\cdot\mathbf{v}(X_c)$ as $X_c(\Omega) \in \mathbb{C}$. Note that $X_c(\Omega)$ is the analytic continuation of $x_c(\omega)$ and $X_c \to x_c$ as $\Omega \to \omega$. We write the Taylor expansion of $M(x)$ around X_c as

$$M(x) = \hat{\Omega} + M_c'\xi + M_c''\frac{\xi^2}{2} + M_c'''\frac{\xi^3}{6} + \dots, \tag{16}$$

where $\xi = x - X_c$ and the derivatives are evaluated at X_c, i.e., $M_c' = M'(X_c)$. Substituting this expression, one of the homogeneous solutions is found to be

$$P^\dagger(x,\Omega) = \xi^3 + \frac{3}{4}\frac{M_c''}{M_c'}\xi^4 + \frac{1}{10}\left(\frac{3}{2}\frac{M_c''^2}{M_c'^2} + 2\frac{M_c'''}{M_c'} + k^2\right)\xi^5 + O(|\xi|^6). \tag{17}$$

The other homogeneous solution includes a logarithmic singularity,

$$P^\ddagger(x,\Omega) = \check{P}^\ddagger(x,\Omega) + A^\dagger(\Omega)P^\dagger(x,\Omega)\mathrm{Log}\,\xi, \tag{18}$$

where

$$\check{P}^\ddagger(x,\Omega) = 1 - \frac{k^2}{2}\xi^2 + O(|\xi|^3), \tag{19}$$

$$A^\dagger(\Omega) = -\frac{k^2 M_c''}{3M_c'}. \tag{20}$$

Notice that, if $A^\dagger(\Omega) = 0$ (or $M_c'' = 0$), the logarithmic term in $P^\ddagger(x,\Omega)$ vanishes and X_c is the apparent singularity [17].

The inhomogeneous terms on the right hand side of (12) requires careful consideration since they also depend on Ω in a complicated manner. By expanding the inhomogeneous terms in series and applying Frobenius's method, the particular solution is solved as

$$P^\sharp(x,\Omega) = \check{P}^\sharp(x,\Omega) + B^\dagger(\Omega)P^\dagger(x,\Omega)\mathrm{Log}\,\xi, \tag{21}$$

where

$$\check{P}^\sharp(x,\Omega) = \tilde{v}|_{\substack{t=0\\x=X_c}}\,\xi + \frac{1}{2}\,(\partial_x\tilde{v} - \tilde{u})|_{\substack{t=0\\x=X_c}}\,\xi^2 + O(|\xi|^3), \tag{22}$$

$$B^\dagger(\Omega) = \frac{1}{3}\left(-i\frac{kM_c'}{c_s}\tilde{p} + \partial_x\tilde{u} - \frac{M_c''}{M_c'}\tilde{u} + k^2\tilde{v}\right)\Bigg|_{\substack{t=0\\x=X_c}}. \tag{23}$$

The logarithmic function $\mathrm{Log}\,(x - X_c)$ has a branch cut on a Riemannian surface. By assuming $\mathbf{k}\cdot\mathbf{v}'(x_c) > 0$, it follows from $\Omega = \mathbf{k}\cdot\mathbf{v}(X_c)$ that, when Ω approaches $\omega \in \sigma_c$ from above or below $\Omega \to \omega \pm i0$, then $X_c \to x_c \pm i0$ and

$$\mathrm{Log}\,(x - X_c) \to \log|x - x_c| \mp i\pi Y(x_c - x), \tag{24}$$

where $Y(x)$ denotes the Heaviside function.

This branch cut of the solution $P(x,\Omega)$ proves the existence of the continuous spectrum σ_c. The inverse Laplace transform whose integral path encircles the continuous spectrum in the counterclockwise direction can be identified as the inverse Fourier transform [8],

$$-\frac{1}{2\pi}\oint P(x,\Omega)e^{-i\Omega t}d\Omega = \int_{\sigma_c}\hat{\tilde{p}}(x,\omega)e^{-i\omega t}d\omega. \tag{25}$$

Therefore, the continuum mode is represented by the Fourier integral of the singular eigenfunctions

$$\begin{aligned} \hat{\tilde{p}}(x,\omega) &:= \frac{1}{2\pi}[P(x,\omega+i0) - P(x,\omega-i0)], \\ &= c^\dagger(\omega)P^\dagger(x,\omega) + c^\ddagger(\omega)\Big[\check{P}^\ddagger(x,\omega) \\ &\quad + A^\dagger(\omega)P^\dagger(x,\omega)\log|x - x_c|\Big] \\ &\quad + c^\sharp(\omega)P^\dagger(x,\omega)Y(x_c - x), \end{aligned} \tag{26}$$

where

$$c^\dagger(\omega) := \frac{1}{2\pi}[C^\dagger(\omega+i0) - C^\dagger(\omega-i0)], \tag{27}$$

$$c^\ddagger(\omega) := \frac{1}{2\pi}[C^\ddagger(\omega+i0) - C^\ddagger(\omega-i0)], \tag{28}$$

$$c^\sharp(\omega) := -\frac{1}{2}[C^\ddagger(\omega+i0) + C^\ddagger(\omega-i0)]iA^\dagger(\omega) - iB^\dagger(\omega). \tag{29}$$

It is important to note that the Heaviside function yields the *third* independent solution in (26). Due to this additional degree of freedom at the singularity, one can compute the singular eigenfunction for every $\omega \in \sigma_c$ by adjusting three coefficients, $c^\ddagger(\omega)$, $c^\ddagger(\omega)$ and $c^\sharp(\omega)$, so that the boundary condition holds.

The existence of point spectra (neutral waves) degenerated in the continuous spectrum is checked by observing $D(\Omega)$ on σ_c. The dispersion relation, $D(\Omega)=0$, has also branch cut on $\omega \in \sigma_c$,

$$D(\omega \pm i0) = D_r(\omega) \pm i\pi A^\dagger(\omega)\partial_x P^\dagger(x_1,\omega)\partial_x P^\dagger(x_0,\omega), \tag{30}$$

where $D_r(\omega)$ is a real function of ω. Since the second term is a pure imaginary function of ω due to the imaginary unit i, both $D(\omega+i0)$ and $D(\omega-i0)$ cannot vanish for real ω unless $A^\dagger(\omega)=0$ (The special case occurs if the regular solution $P^\dagger(x,\omega)$ happens to satisfy both boundary conditions; $\partial_x P^\dagger(x_1,\omega)=0$ and $\partial_x P^\dagger(x_0,\omega)=0$. But, it seems to be very rare). For a point spectrum, say ω_0, to occur in the continuous spectrum, the condition $A^\dagger(\omega_0)=0$ must be satisfied, namely, the corresponding resonant point x_c must be an inflection point of the basic flow along the wavevector $\mathbf{k}$ (so-called generalized inflection point);

$$\mathbf{k}\cdot\mathbf{v}''(x_c) = 0. \tag{31}$$

4. Asymptotic behavior

Once we get the solution $P(x,\Omega)$ in the form of (13), the other dependent variables are derived immediately by

$$V(x,\Omega) = -i\frac{\partial_x P - \tilde{v}|_{t=0}}{\Omega - \mathbf{k}\cdot\mathbf{v}}, \tag{32}$$

$$U(x,\Omega) = \frac{\mathbf{k}\cdot\mathbf{v}'V + ik^2P + i\,\tilde{u}|_{t=0}}{\Omega - \mathbf{k}\cdot\mathbf{v}}, \tag{33}$$

$$W(x,\Omega) = \frac{(\mathbf{k}\times\mathbf{v}')_x V + i\,\tilde{w}|_{t=0}}{\Omega - \mathbf{k}\cdot\mathbf{v}}. \tag{34}$$

If there is a point spectrum $\omega_0 \in \sigma_p$ which appears as a pole $1/(\Omega-\omega_0)$ in $C^\dagger(\Omega)$ and $C^\ddagger(\Omega)$, the inverse Laplace transform gives the exponential behavior ($e^{-i\omega_0 t}$) of the corresponding eigenfunction as usual.

The initial value problem is completed by taking account of the continuous spectrum. Let us focus on the behavior (25) stemming from the continuous spectrum. The asymptotic behavior is roughly estimated by the highest order of singularity in $P(x,\Omega)$. We have seen that the singularity of $P(x,\Omega)$ on σ_c is not so strong and the phase mixing results in damping of the continuum mode;

$$P(x,\Omega) \sim \xi^3 \mathrm{Log}\,\xi \;\Rightarrow\; \tilde{p}(x,t) \sim t^{-4}e^{-i\mathbf{k}\cdot\mathbf{v}(x)t}. \tag{35}$$

By substituting (13) into (32) and (33), some eliminations of terms occur in the lowest order of ξ and the division by $\Omega - \mathbf{k}\cdot\mathbf{v}$ does not create the singularity of $1/\xi$ in $V(x,\Omega)$ and $U(x,\Omega)$. The asymptotic behavior is estimated as

$$V(x,\Omega) \sim \xi \mathrm{Log}\,\xi \;\Rightarrow\; \tilde{v}(x,t) \sim t^{-2}e^{-i\mathbf{k}\cdot\mathbf{v}(x)t}, \tag{36}$$

$$U(x,\Omega) \sim \mathrm{Log}\,\xi \;\Rightarrow\; \tilde{u}(x,t) \sim t^{-1}e^{-i\mathbf{k}\cdot\mathbf{v}(x)t}. \tag{37}$$

The algebraic power of the damping (36) is same as the incompressible case studied by Eliassen *et al.* [11]. On the other hand, $W(x,\Omega)$ of (34) includes the singularity of $1/\xi$, which is associated with another continuous spectrum residing in the Squire part. This singularity yields the vortical continuum mode that never damps;

$$W(x,\Omega) \sim \frac{1}{\xi} \;\Rightarrow\; \tilde{w}(x,t) \sim \tilde{w}(x,0)e^{-i\mathbf{k}\cdot\mathbf{v}(x)t}, \tag{38}$$

where we have also indicated the amplitude since it is easily obtained from the inverse Laplace transform of $i\,\tilde{w}|_{t=0}/(\Omega - \mathbf{k}\cdot\mathbf{v})$. Although there are two kinds of continuous spectra that are completely overlapping with each other, no algebraic growth occurs due to the phase mixing damping of $\tilde{v}(x,t)$.

However, the estimation of the asymptotic behavior will be modified when some point spectra exist inside the continuous spectrum. As was shown in the previous section, such a degenerated point spectrum $\omega_0 \in \sigma_c$ can occur if the condition (31) is satisfied at $x_c(\omega_0)$ and both $P^\dagger(x,\omega_0)$ and $P^\ddagger(x,\omega_0)$ are regular (the apparent singularity). While the logarithmic singularity still exists in $P^\sharp(x,\omega_0 \pm i0)$, the pole $C^\dagger(\Omega), C^\ddagger(\Omega) \propto 1/(\Omega-\omega_0)$ brings about a neutral oscillation $e^{-i\omega_0 t}$ in the Rayleigh part $(\tilde{v},\tilde{u},\tilde{p})$. The continuum mode in the Squire part is now forced by this eigenmode. Indeed, $W(x,\Omega)$ in (34) includes the following strong singularity,

$$W(x,\Omega) \sim \frac{1}{(\Omega-\omega_0)(\Omega-\mathbf{k}\cdot\mathbf{v})} \;\Rightarrow\; \tilde{w}(x,t) \sim \frac{e^{-i\omega_0 t} - e^{-i\mathbf{k}\cdot\mathbf{v}(x)t}}{\omega_0 - \mathbf{k}\cdot\mathbf{v}(x)}, \tag{39}$$

unless $(\mathbf{k}\times\mathbf{v}')_x(x_c)=0$ at the resonant point $x_c(\omega_0)$. This shows that $\tilde{w}(x,t)$ grows linearly only at $x_c(\omega_0)$, where the continuum mode is locally forced by the neutral wave in analogy with the Jordan block.

5. Stability of plane Couette flow (Linear shear flow)

In order to demonstrate the algebraic instability caused by neutral acoustic modes, the plane Couette flow is a simplest example, where the shear profile is linear $v_y(x), v_z(x) \propto x$ and hence the condition (31) is satisfied everywhere.

In this section, we regard the domain as $[x_0,x_1]=[-1,1]$ and introduce a parameter $\alpha \in \mathbb{R}$ by

$$M(x) = \frac{\mathbf{k}\cdot\mathbf{v}(x)}{kc_s} = \alpha x, \tag{40}$$

where the Galilei transformation of the coordinates allows us to assume $M(0)=0$ without loss of generality. The parameter α basically measures the shear of the flow along $\mathbf{k}$. In this case, the continuous spectrum extends over

$$\sigma_c = \{\omega \in \mathbb{R}; -\alpha \le \hat{\omega} \le \alpha\}, \tag{41}$$

where $\hat{\omega}=\omega/kc_s$, and the resonant point is simply $x_c = \hat{\omega}/\alpha$. The calculations in the previous sections are greatly simplified by the fact that $A^\dagger(\Omega)\equiv 0$. The two homogeneous solutions, $P^\dagger(x,\Omega)$ and $P^\ddagger(x,\Omega)$, then turn out to

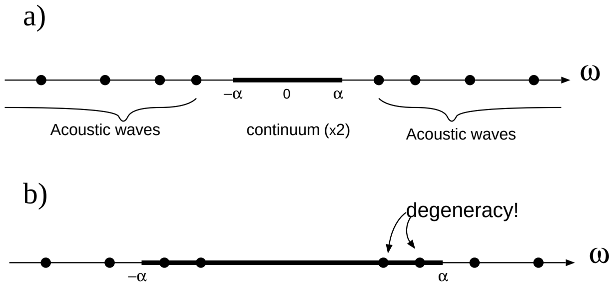

Figure 2: Spectrum of disturbances in shear flow.

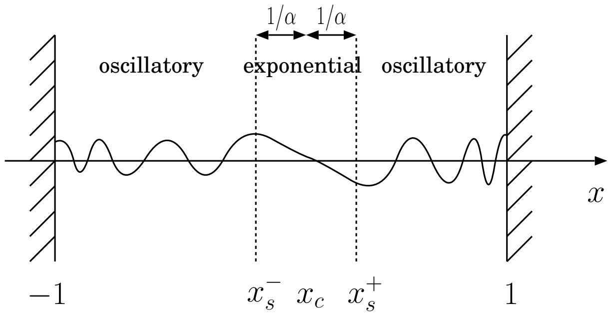

Figure 3: Schematic view of the acoustic mode $P(x, \omega_n)$ with the eigenvalue ω_n degenerated into the continuous spectrum σ_c.

be regular, whereas the logarithmic singularity still exists in the particular solution $P^\sharp(x, \Omega)$.

Let us investigate the existence of point spectra on the real axis $\omega = \mathrm{Re}(\Omega)$. Recall that $P^\dagger(x, \omega)$ and $P^\ddagger(x, \omega)$ are the solutions of the homogeneous part of the equation (12), which is now written in the Sturm-Liouville type,

$$(42)\ \partial_x \left[\frac{1}{(\hat{\omega} - \alpha x)^2} \partial_x P\right] - k^2 \left[\frac{1}{(\hat{\omega} - \alpha x)^2} - 1\right] P = 0,$$

where the resonant point $x_c = \hat{\omega}/\alpha$ is the apparent singularity. If some linear combination of $P^\dagger(x, \omega)$ and $P^\ddagger(x, \omega)$ satisfies the boundary condition for a certain value $\omega_0 \in \mathbb{R}$, then it is an eigenvalue $D(\omega_n) = 0$.

We can apply the Sturm's oscillation theorem [17] to (42). For $\omega \in \mathbb{R}$ being sufficiently apart from the continuous spectrum, the point x_c does not appear in the domain $[-1, 1]$ and the spatial structures of $P^\dagger$ and $P^\ddagger$ are oscillatory. The Sturm theorem prove the existence of infinite number of point spectra $\sigma_p = \{\pm\omega_n\}$ accumulating to the infinity $|\omega_n| \to \infty$ (see Figure 2a). In the case of $\alpha = 0$, these ω_n's are given by

$$(43)\ \hat{\omega}_n = \frac{\omega_n}{k c_s} = \sqrt{\frac{n^2\pi^2}{4k^2} + 1} \quad (n = 0, 1, 2, \dots),$$

representing the acoustic modes.

Some of these point spectra may degenerate into the the continuous spectrum when the shear gradient is sufficiently steep. When α increases and exceeds 1, the continuous spectrum expands and eventually overlaps the point spectra, $\omega_0, \omega_1, \dots,$ as illustrated in Figure 2b. The acoustic modes corresponding to such degenerated point spectra have the resonant point x_c in the domain. The schematic view of such a mode $P(x, \omega_n)$ is illustrated in Figure 3. The two positions $x_s^\pm = x_c \pm 1/\alpha$ are called sonic points. The mode structure becomes evanescent (non-oscillatory) in the interval $[x_s^-, x_s^+]$, in which the phase speed of the wave is subsonic in comparison with the basic flow.

One might think that this degeneracy of the point-continuous spectra contradicts the physical picture of the eikonal approximation ($k \to \infty$), in which the ray of any acoustic wave must be reflected at the sonic points (see Appendix A). However, if k is finite, the wave can penetrate the evanescent region like the tunneling effect.

We can show this degeneracy by making use of the Sturm theorem again. If the two linearly independent solutions, $P^\dagger$ and $P^\ddagger$, are given for some α and k, they have a scaling symmetry

$$(44)\qquad P^{\dagger,\ddagger}(x, \omega; \alpha, k) = P^{\dagger,\ddagger}(x/s, \omega; s\alpha, sk),$$

for arbitrary number $s \in \mathbb{R}$, which follows from the same symmetry of the equation (42). Let us fix $\omega \in \sigma_c$ such that the point x_c appears in the domain. By increasing α and k simultaneously ($s \nearrow \infty$), the spatial structures of the solutions will uniformly shrink. Since the solutions are oscillatory in the outside of $[x_s^-, x_s^+]$, the numbers of their nodes in the domain will monotonically increase. Thus, in the same manner as the Sturm's oscillation theorem, we get an infinite number of characteristic values of (α, k) at which a linear combination of the solutions satisfies the boundary condition and the prescribed ω becomes the eigenvalue of (42).

We solved the eigenvalue problem (42) by the numerical integration. In Figure 4, the real and imaginary parts of the (normalized) eigenvalues $\hat{\omega}_n$ are shown with respect to the flow shear α for fixed $k = 1$. There are some exponential instabilities for large α, which are essentially variants of the unstable acoustic modes found by Mack [14] in the boundary layer. It is remarkable that these unstable modes remain present in the plane Couette flow where the Kelvin-Helmholtz instability (or the first mode called by Mack) is definitely absent. Our result shows that the linear shear flow can destabilize the neutrally stable eigenmodes by inducing the modal coupling among them.

We have drawn a dotted line $\mathrm{Re}(\hat{\omega}) = \alpha$ in Figure 4 to indicate that the right side of this line is covered by the continuous spectrum σ_c. The eigenvalues $\{\hat{\omega}_n\}$ of the neutral acoustic modes indeed enter the continuous spectrum one by one as α increases. We have shown that these neutral eigenmodes respectively excite the vortical continuum modes in the Squire part locally at their resonant points $x_c(\omega_n)$.

The exponential instability first emerges at a critical Mach number α_e ($\alpha_e \simeq 4.2$ for $k = 1$ as shown in Figure 4). Below the critical value α_e, the degeneracy of ω_0

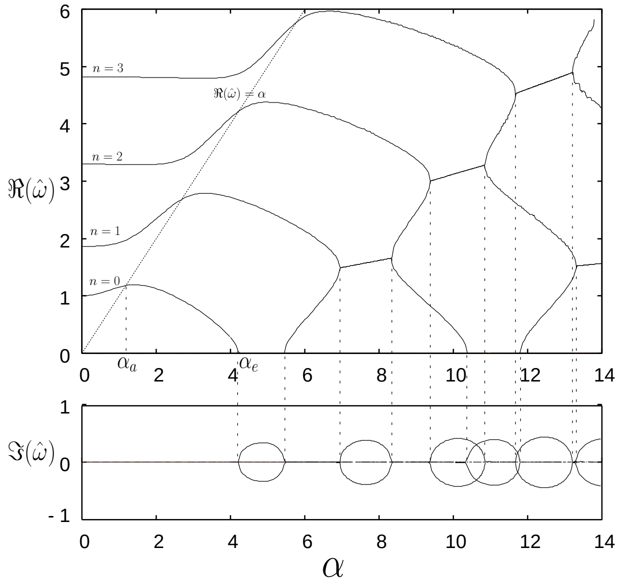

Figure 4: Eigenvalue diagram $\hat{\omega}_n - \alpha$ of acoustic modes for fixed $k = 1$. The right hand side of the dashed line $\mathrm{Re}(\hat{\omega}) = \alpha$ is filled with the continuous spectrum $\sigma_c = [-\alpha, \alpha]$.

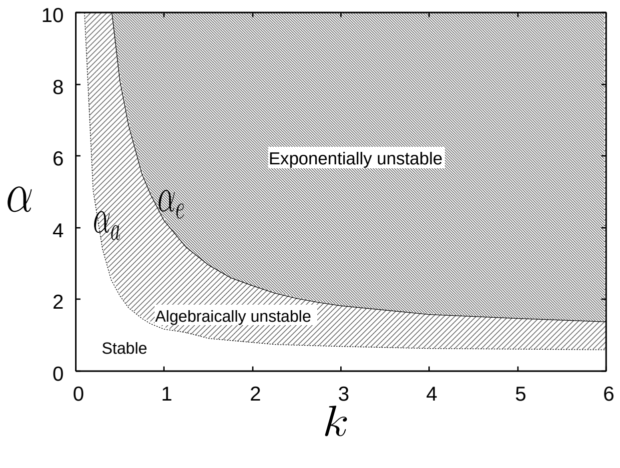

Figure 5: Critical values of α [defined by (40)] with respect to wavenumber k; the algebraic and exponential instabilities emerge respectively at α_a and α_e.

into the continuous spectrum triggers the algebraic instability for α greater than α_a ($\alpha_a \simeq 1.17$ for $k = 1$). We numerically calculated α_a and α_e for various k, which are plotted in Figure 5. This result shows that the values of α_a and α_e respectively approach 0.5 and 1 as $k \to \infty$. It follows that the algebraic instability is dominant in plane Couette flows for $0.5 < \alpha < 1$.

6. Summary

The spectrum of disturbances in a bounded shear flow is composed of two continuous spectra (vortical continuum modes) and infinite number of point spectra (acoustic modes). The acoustic modes can interact with the continuum modes when the shear flow is supersonic and the point spectra of the acoustic modes are degenerated into the continuous spectrum. The resonant interaction among these spectra is due to the non-selfadjoint property of the linearized system. We have found an algebraic instability arising from this resonant interaction by means of the Laplace transform approach.

This degeneracy occurs only when the second derivative of the flow profile along the wavenumber vector is zero, $\mathbf{k}\cdot\mathbf{v}''(x_c) = 0$, at the resonant point x_c (or the critical layer) of the acoustic mode, as was already proven by Lees and Lin [13]. We showed that the degeneracy between point and continuous spectra is analogous to the Jordan block and the resonance between them causes algebraic growth of the continuum mode (which corresponds to the normal vorticity $\tilde{w}$ in the Squire part) that is spatially localized to x_c and temporally in proportion to time.

The plane Couette flow is a simplest example, where the condition $\mathbf{k} \cdot \mathbf{v}''(x_c) = 0$ is satisfied everywhere for any $\mathbf{k}$. The existence of the degenerated point spectra is confirmed both analytically and numerically. While some exponentially unstable modes arise from the coupling between point spectra, the algebraic instability occurs even in the subcritical cases of the exponential instability. In particular, the algebraic instability is dominant in a bounded channel $[-1, 1]$ when the variation of velocity shear normalized by the sound speed, i.e., α in (40), is between 0.5 and 1.

Acknowledgments

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A. Ray tracing of acoustic waves

We consider the propagation of the acoustic waves in a shear flow $\mathbf{v}(x) = (0, v_y(x), v_z(x))$. Suppose the wavenumber $\mathbf{k} = (k_z, k_y, k_z)$ is sufficiently large compared to the gradient of the ambient shear ($|\mathbf{k}| \gg |\mathbf{v}'(x)|$). We may assume, due to the eikonal approximation, that the acoustic waves *locally* obey the dispersion relation $\omega(\mathbf{x}, \mathbf{k}) =$

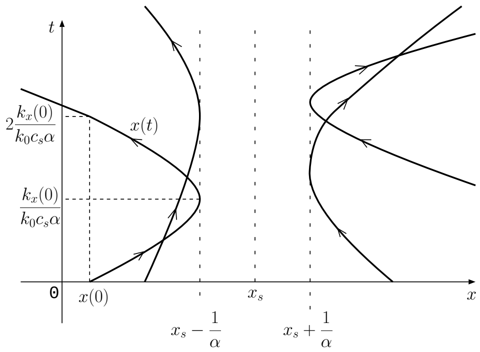

Figure 6: Rays of acoustic waves.

$\mathbf{k}\cdot\mathbf{v}(x) \pm kc_s$, where $k = |\mathbf{k}|$, at each position $\mathbf{x} = (x, y, z)$. The rays of the acoustic waves are governed by the eikonal equations for $\mathbf{x}(t)$ and $\mathbf{k}(t)$,

$$\partial_t \mathbf{x} = \frac{\partial \omega}{\partial \mathbf{k}} = \begin{pmatrix} 0 \\ v_y(x) \\ v_z(x) \end{pmatrix} \pm \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \frac{c_s}{k}, \tag{45}$$

$$\partial_t \mathbf{k} = -\frac{\partial \omega}{\partial \mathbf{x}} = \begin{pmatrix} -\mathbf{k}\cdot\mathbf{v}'(x) \\ 0 \\ 0 \end{pmatrix}. \tag{46}$$

For simplicity, let us assume linear shear flow. Since the latter equation gives $k_y = $ const. and $k_z = $ const., we may introduce a constant α by $\mathbf{k}\cdot\mathbf{v}(x)/k_0 c_s = \alpha x$ where $k_0 = (k_y^2 + k_z^2)^{1/2}$. Then, the equations are solved as

$$k_x(t) = k_x(0) - k_0 c_s \alpha t, \tag{47}$$

$$x(t) = \pm\frac{k(0) - k(t)}{k_0 \alpha} + x(0) = \mp\frac{k(t)}{k_0\alpha} + x_c, \tag{48}$$

where $k(t) = \sqrt{k_x(t)^2 + k_y^2 + k_z^2}$ and $x_c = \omega/k_0 c_s \alpha$. Note that $\omega = \alpha x(0) \pm k(0) c_s$ is a conserved quantity along the ray. When $t = k_x(0)/k_0 c_s \alpha$, the wavenumber $k(t)$ takes a minimum k_0. For fixed k_y, k_z and ω, one can locate three points

$$x_c - \frac{1}{\alpha}, \quad x_c, \quad x_c + \frac{1}{\alpha}, \tag{49}$$

on the x axis. Some orbits of $x(t)$ with different initial data $x(0)$ and $k_x(0)$ are illustrated in Figure 6. One can see that all rays of acoustic waves are reflected at $x = x_c \pm \alpha^{-1}$ and cannot propagate the interval $[x_c - \alpha^{-1}, x_c + \alpha^{-1}]$, that is, *cutoff* or *evanescent* region. The eikonal approach, therefore, cannot construct any global eigenmode beyond this interval.

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