Journal of Math-for-Industry, Vol. 5 (2013B-8), pp. 145–149

Tropical geometry of PERT

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Received on March 18, 2013 / Revised on September 7, 2013

Abstract. Based on a description of project networks by max-plus algebra and poset, the adjacency of critical paths is presented using tropical geometry.

Keywords. tropical geometry, discrete event system, CPM

1. Introduction

The max-plus algebra, also known as the tropical semiring, appears in discrete event systems [1]. For instance, the event firing time of a Petri net is analysed by max-plus linear algebra. PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are popular methods of scheduling, and the problem of resource conflict is discussed in [4].

Suppose there is a set of several activities with an order; each activity can be started after all the preceding activities have accomplished. The activities form a *project network*. Each activity x is assigned with a nonnegative number t_x representing the time required for x, which is called the *time cost* of x. One usually includes the start activity u and the end activity y both with time cost zero.

Fix a project network. A sequential activities from u to y is called a *path* unless otherwise noted. For the accomplishment of a path I, we need at least the sum of the time costs along I. Take the maximum for all the paths, and you get the *earliest finishing time* for the project itself. The path attaining the maximum is called a *critical path.*

Since the critical path determines the time necessary to complete all the activities, it is important to watch and control the critical path. If one can reduce the time cost of an activity on the critical path, the total duration becomes shorter. On the contrary, an accident on the critical path directly results in the delay of the closing.

However, if an activity out of the critical path suddenly requires more time, or some activities in the critical path reduces their time costs, the critical path might change. If so, which path is likely to be critical? In this paper, the question is answered geometrically by the adjacency of paths. Example 3 illustrates how one can visualize the possible transition of paths.

We formulate a project network as a graph and an ordered set in Section 2, and show when a tropical polynomial is realised as the earliest finishing time in Section 3. The topology of the transition of critical paths is discussed in Sections 4 and 5.

2. Graph representation and poset structure of a project network

A project network is usually represented by a directed graph called a *PERT chart*, and there are two popular ways; namely, *activity-on-arrow* (AOA) and *activity-on-node* (AON) diagrams. In the AOA diagram, each activity is represented by an arrow (a directed edge). A vertex represents a milestone between activities. On the other hand, in the AON diagram, an activity is represented by a vertex and a dependency between activities by an arrow. Both diagrams can represent any project networks, but we adopt the AON diagram, where the insertion of dummy arrows (e.g. the dotted arrow in Figure 1, with no corresponding activities) are not needed.

Every AON diagram is *simple*, that is, has no self-loops or multiple arrows. Thus an arrow is represented by a pair of two different (namely, initial and terminal) vertices. By definition there are no circular references, so the graph contains no cycles. We may assume that an arrow connects only a pair of immediate predecessor-successor (i.e. there are no activities in between); e.g. if $x_0 \to x_1 \to \cdots \to x_k$ ($k \geq 2$) exists then we do not include the arrow $x_0 \to x_k$. This condition is natural because time cost of the former path is no less than that of the latter path because all time costs are semipositive. We call that final condition as having no '*short-cuts*'.

We shall identify such a graph and a partially ordered set ('poset') as follows. X is supposed to be the set of activities.

Proposition 1. *Let X be a finite set. There is a one-to-one correspondence between the set of partial orders on X and the set of simple directed graphs with vertex set X without cycles or short-cuts.*

Proof. First note that for a finite ordered set, $x < x'$ is equivalent to the existence of a finite sequence of immediate predecessor-successors $x = x_0 < \cdots < x_k = x'$. This can be done by inductive insertions of intermediate elements if exist.

Activity	Predecessor
a	-
b	-
c	a
d	a, b

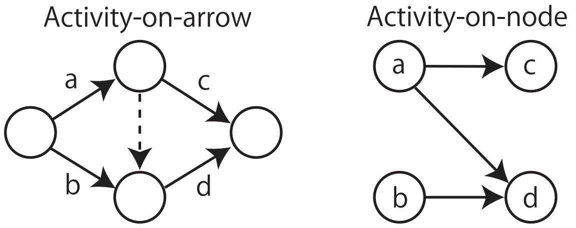

Figure 1: PERT charts (u and y are omitted)

Assume a partial order is given to X. Combining an immediate predecessor and an immediate successor by an arrow, one get a simple directed graph without short-cuts. The antisymmetric law guarantees the vacancy of cycles. The graph is nothing but a Hasse diagram.

On the contrary, from a given directed graph, the reachability along a finite (≥ 0) sequence of arrows defines a partial order. The reflectivity and transitivity is obvious. $x < x'$ and $x' < x$ does not occur simultaneously since there are no cycles.

It is easy to see that the compositions of those two maps in both directions are identity maps. $\square$

Thus, a PERT chart without short-cuts is equivalent to a Hasse diagram of a poset with a minimum element u and a maximal element y such that the vertices are nonnegatively weighted. A path is a graph-theoretic path from u to y in the Hasse diagram, which constitutes a totally-ordered set of vertices. Assumption of 'no short-cuts' guarantees that the totally-ordered set is maximal; if one can add another activity, the insertion must be somewhere in-between, thus a short-cut exists. On the contrary, suppose a poset X with u and y is given. Take a maximal totally-ordered subset I. Since I is maximal, I contains u and y, and the covering relation on I extends to the immediate predecessor-successor relation on X. Thus I corresponds to a path of the PERT chart.

We omit u and y from X unless necessary from now on. The following remark holds by considering topological ordering.

Remark 1. Every finite poset $(X, \leq)$ can be embedded into a totally-ordered set $(X, \leq')$ with the same set.

By the remark, we may always assume that a finite poset of order n is equivalent to $[n] := \{1, 2, \ldots, n\}$ as a set, where $i < j$ in the poset order implies $i < j$ as integers. The converse implication is not necessarily true. A totally-ordered set corresponds to a serial PERT chart, whereas in the case of a parallel PERT chart, as a poset $[n]$ is an *antichain*, where distinct elements are not comparable.

3. Earliest finishing time as a tropical polynomial

Let F be the earliest finishing time of a project network with n activities. We denote the time costs by $t_1, \ldots, t_n$. Since the change of states matters, the time costs are treated as variables, not as constants.

For a subset I of $[n]$, we write the monomial term $\prod_{i \in I} t_i$ as t_I.

Proposition 2. *The earliest finishing time F can be written as a tropical polynomial of $t_1, \ldots, t_n$ satisfying the following three conditions:*

1. *the degree of F on each variable is at most one,*
2. *the coefficient of each term is a unit,*
3. *no term is divisible by any other terms. ('nondivisibility')*

Proof. For each path, the sum of the time costs is a tropical monomial, and the maximum is represented by the tropical addition. Condition 1 follows since every path contains each activity at most once. Condition 2 is obvious. Thus every term of F can be written as t_I for some I, where I is a maximal totally-ordered subset of $[n]$. Suppose a term t_J is divisible by a term t_I. This is equivalent to saying that J contains I. Since I is maximal, $I = J$ follows. $\square$

Let us think of the inverse problem: what is a sufficient condition for a tropical polynomial to have a corresponding PERT chart?

We call a nonconstant tropical polynomial F *prerealisable* if F satisfies the three conditions of Proposition 2. Prerealisable polynomial can be written in the form of $\sum_{I \in \mathcal{I}} t_I$, where $\mathcal{I}$ is a nonempty subset of the power set $2^{[n]}$ which satisfies that if $I, J \in \mathcal{I}$ and $I \subset J$ then $I = J$. Moreover, we say that a tropical polynomial is *PERT-realisable* (or simply, *realisable*) if there exists a PERT chart with the earliest finishing time being F. By Proposition 2, a realisable tropical polynomial is prerealisable.

Proposition 3. *Let $F = \sum_{I \in \mathcal{I}} t_I$ be a tropical polynomial of n variables. Then F is realisable iff there exists a poset structure on the index set $[n]$ such that*

$$I \text{ is a maximal totally-ordered subset} \iff t_I \text{ is a term of } F.$$

Proof. If F is realisable, then we have the equivalence since they both mean that I is a path of the PERT chart. Suppose that the index set has a poset structure satisfying the given equivalence. Then from Proposition 1, we have a corresponding PERT chart such that every maximal totally-ordered subset corresponds to a path and vice versa. Thus F is the earliest finishing time of the PERT chart. $\square$

Prerealisable polynomials are not always realisable as shown below.

Proposition 4. *Let $\mathcal{I}$ be a subset of $2^{[n]} \setminus \{[n]\}$ such that for every $i, j \in [n]$, there exists $I \in \mathcal{I}$ including both i and j. Then $F = \sum_{I\in\mathcal{I}} t_I$ is not realisable.*

Proof. Suppose F is realisable. Then $[n]$ has a poset structure $\leq$ such that every $I \in \mathcal{I}$ is totally ordered. Then $([n], \leq)$ is totally ordered since every two elements are comparable. This contradicts to the maximality of I. □

Let n, k be positive integers satisfying $k \leq n$. Take $\mathcal{I}$ to be $\{I \subset [n] \mid \#I = k\}$. We write the corresponding tropical polynomial F as $F_{n,k}$. Apparently $F_{n,k}$ is prerealisable.

Corollary 1. *$F_{n,k}$ is realisable iff $k = 1$ or n.*

Proof. If $k = 1$ (resp. $k = n$), then $F_{n,k}$ corresponds to the PERT chart consisting of n parallel (resp. serial) activities. On the other hand, if $2 \leq k \leq n-1$, $F_{n,k}$ is not realisable by the previous proposition. □

A simple nonrealisable example is $F_{3,2} = t_1t_2 \oplus t_2t_3 \oplus t_1t_3$.

4. Tropical hypersurface and critical paths

We consider the topology of the space of paths. First we fix the notation.

We regard $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ as the tropical coordinates of $\boldsymbol{R}^n$ equipped with Euclidean topology, though we mainly work on the semipositive orthant $\boldsymbol{R}^n_{\geq 0}$, which we denote by Γ. For $I \subset [n]$, we denote by $\boldsymbol{e}_I$ the element of Γ such that t_i equals one for $i \in I$ and zero otherwise. When $I = \{i\}$, we write $\boldsymbol{e}_{\{i\}}$ as $\boldsymbol{e}_i$.

For a tropical polynomial $F = F(t_1, \ldots, t_n)$, the tropical hypersurface $V(F)$ in $\boldsymbol{R}^n$ is the locus where more than one terms attain the maximum. We denote by $V(F)_{\geq 0}$ the intersection $V(F) \cap \Gamma$ and by Γ_F the complement $\Gamma \setminus V(F)_{\geq 0}$.

In the sequel, fix a tropical polynomial $F = \sum_{I\in\mathcal{I}} t_I$ and $I \in \mathcal{I}$. We define a close (resp. an open) convex polyhedral cone C_I (resp. C_I°) to be $\bigcap_{J\in\mathcal{I}}\{\boldsymbol{t} \in \Gamma \mid t_I \geq t_J\}$ (resp. $\bigcap_{J\in\mathcal{I}}\{\boldsymbol{t} \in \Gamma \mid t_I > t_J\}$). Since $V(F)_{\geq 0} = \bigcup_{\substack{J,K \\ J\neq K}} (C_J \cap C_K)$, $V(F)_{\geq 0}$ is also a cone.

Proposition 5. *If F is prerealisable, the following statements hold:*

1. *C_I° is a nonempty and C_I is n-dimensional.*
2. *C_I° is the interior of C_I in Γ.*
3. *C_I is the closure of C_I° in Γ.*

Proof. We first prove statement 1. The value of t_I at $\boldsymbol{e}_I$ equals $\#I$, and the value of a term t_J at $\boldsymbol{e}_I$ is $\#(I \cap J)$. Thus t_I is strictly the biggest term of F at $\boldsymbol{e}_I$ by nondivisibility, thus $\boldsymbol{e}_I \in C_I^\circ$. C_I contains an nonempty open subset C_I°, thus is n-dimensional.

We now prove statement 2. Suppose $\boldsymbol{t} \in C_I$ satisfies $t_I = t_J$ for some distinct I, J in $\mathcal{I}$. By nondivisibility, there exists $j \in J \setminus I$. Then for any $\varepsilon > 0$, $\boldsymbol{t} + \varepsilon\boldsymbol{e}_j$ is in Γ but not in C_I since $t_I < t_J$. This shows that $\boldsymbol{t}$ is a boundary point of C_I in Γ, and the interior of C_I in Γ is contained in C_I°. C_I° is open in Γ, thus is the interior of C_I.

For statement 3, let $\boldsymbol{t}$ be a point in C_I. For any $\varepsilon > 0$, $\boldsymbol{t} + \varepsilon\boldsymbol{e}_I$ is in Γ and belongs to C_I° by nondivisibility. Thus $\boldsymbol{t}$ is in the closure of C_I°. Since C_I is closed, it coincides to the closure of C_I°. □

A prerealisable polynomial has a remarkable characterization as a function on the semipositive orthant.

Proposition 6. *A tropical polynomial $F = \sum_{I\in\mathcal{I}} t_I$ satisfies the nondivisibility condition iff each t_I solely attains the maximum at some point in Γ.*

Proof. If F satisfies the nondivisibility condition, then C_I° is nonempty for all I from the previous proposition. On the contrary, suppose $I \subset J$, where $I, J \in \mathcal{I}$. Then $\{\boldsymbol{t} \in \Gamma \mid t_I > t_J\}$ is an empty set, and we have $C_I^\circ = \emptyset$. □

Theorem 1. *Let F be the earliest finishing time of a project network. Then $V(F)_{\geq 0}$ is the set of points in Γ which has two or more critical paths. The points in each connected component of Γ_F has the same unique critical path. Every path becomes a critical path for some time costs.*

Proof. By the definition of critical path and tropical hypersurface, $\boldsymbol{t} \in V(F)$ is equivalent to that there are more than one critical paths for the time cost $\boldsymbol{t} \in \Gamma$.

We show by prerealisability that C_I° is a connected component of Γ_F and $\Gamma_F = \bigcup_{I\in\mathcal{I}} C_I^\circ$. Tropical monomials are linear functions, hence continuous. Thus, the difference of any two tropical monomials in F has a same sign on each connected component U of Γ_F. Hence the maximum term is the same on U and U is contained in some C_I°. Since C_I° is convex, if at $\boldsymbol{t}$ and $\boldsymbol{t}'$ the tropical polynomial F attains a same sole maximum term t_I, the points are connected by a segment in C_I°.

The last statement follows from the nonemptiness of C_I°. □

Remark 2. If we allow the time costs to take negative values, the previous theorem still remains valid. The general version may be useful for possible generalizations like this. In the problem of percolation or invasion, the AND condition 'all the previous activities is finished' in PERT is replaced to the OR condition 'at least one previous activity is finished'. In this case we have the MIN-plus algebra instead. By multiplying (-1) to all the time costs, one can regard the problem as a negative-time-costs version of PERT.

5. Adjacency of paths

When the time costs change, on purpose or by accident, a change of criticality of paths may follows. Let us define a graph to monitor the transition.

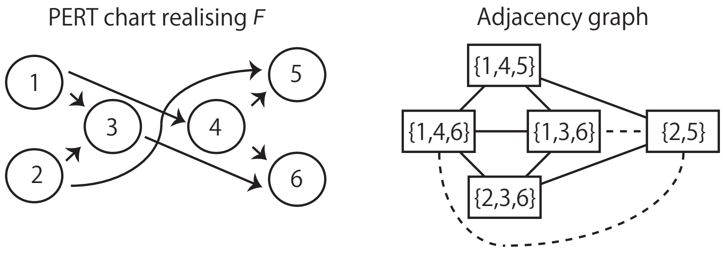

Figure 2: $N(F) \neq G(F)$

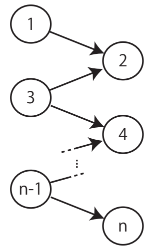

Figure 3: Zigzag PERT chart

Take two maximal paths I, J in a PERT chart with the set of paths $\mathcal{I}$, or in general, two monomial terms t_I, t_J in a prerealisable polynomial F. We say C_I, C_J are *adjacent* (or I, J are *adjacent*), if C_I, C_J have a common codimension-one wall. When criticality of a path changes, some of the adjacent paths always attain the maximum (at least in a moment).

Let $G(F)$ be an undirected graph with the vertex set $\mathcal{I}$ and the edges between adjacent vertices. We call $G(F)$ the *adjacency graph* of F. By Proposition 6, we identify each vertex I with the chamber C_I.

Example 1. $F_{n,1} = t_1 \oplus \cdots \oplus t_n$ in Corollary 1 is realisable as a parallel chart. For every $i, j \in [n]$, $C_{\{i\}}$ and $C_{\{j\}}$ are adjacent around a point $\boldsymbol{e}_{\{i,j\}}$ along the codimension-one wall $\{t_i = t_j\}$. Thus we have $G(F_{n,1}) = K_n$, the complete graph with n vertices.

For $2 \leq k \leq n-1$, chambers C_I, C_J of $F_{n,k}$ are adjacent iff $\#(I \cap J) = k-1$, which is because $C_I = \bigcap_{i \in I,\, j \in [n] \setminus I} \{\boldsymbol{t} \in \Gamma \mid t_i \geq t_j\}$.

Example 2. Let $F = \sum_{i=1}^{n-1} t_i t_{i+1}$ $(n \geq 2)$. Then $G(F)$ is K_{n-1}, again a complete graph. Note that F corresponds to the PERT chart illustrated at Figure 3.

For a prerealisable tropical polynomial $F = \sum_{I \in \mathcal{I}} t_I$, we define the Newton polytope of F to be the convex hull of $\{\boldsymbol{e}_I \mid I \in \mathcal{I}\}$. Since all the coefficients of F is a unit, this polytope coincides with the Newton subdivision defined by canonically projecting the codimension-two skeleton of $\{(\boldsymbol{e}_I, \alpha) \in \boldsymbol{R}^n \times \boldsymbol{R} \mid I \in \mathcal{I},\ \alpha \leq$ coefficient of the term of which multidegree is $I\}$ to $\boldsymbol{R}^n$ by deleting the last coordinate.

On the other hand, it is well-known (see, e.g., [3]) that the Newton subdivision of F is dual to $V(F)$, i.e. there is a one-to-one correspondence between a cell of the Newton subdivision and a polytope of $V(F)$ of complementary dimension which is orthogonal to the cell (this duality holds without the assumption of prerealisability of F). Particularly for the above F, each cell of the Newton subdivision (which is same as the Newton polynomial) given as the convex hull of $\boldsymbol{e}_{I_1}, \ldots, \boldsymbol{e}_{I_k}$ corresponds to the polytope of $V(F)$ such that $t_{I_1}, \ldots, t_{I_k}$ attain the same maximal value.

Let $N(F)$ be the 1-skeleton of the Newton subdivision of F. Then for each chamber $C_I \subset \Gamma$, the corresponding vertex of $G(F)$ is I. On the other hand, the chamber of $\boldsymbol{R}^n \subset V(F)$ containing C_I is dual to the the vertex $\boldsymbol{e}_I$ of $N(F)$. If C_I and C_J are adjacent, then there exists an edge of $G(F)$, and also an edge connecting $\boldsymbol{e}_I$ and $\boldsymbol{e}_J$ of $N(F)$. Thus $G(F)$ is a subgraph of $N(F)$.

Since the number of the vertices of $G(F)$ equals $\#\mathcal{I}$, the number of vertices of $G(F)$ and $N(F)$ coincides. However, some edges in $N(F)$ may not appear in $G(F)$.

Example 3. Let F be a prerealisable tropical polynomial

$$F = t_1 t_3 t_6 \oplus t_1 t_4 t_5 \oplus t_1 t_4 t_6 \oplus t_2 t_3 t_6 \oplus t_2 t_5.$$

Then F is realisable, since there exists a corresponding PERT chart drawn left side of Figure 2. The right figure is $G(F)$, where boxes represents vertices with I written inside, and the the bold lines are the edges (which was obtained by calculating $V(F)$). However, $N(F)$ contains the dotted lines too. In fact, $G(F)$ cannot be obtained as the 1-skeleton of a Newton subdivision of any prerealisable polynomial. This is because for any polytope of dimension d, the degree of each vertex must be bigger than d. Thus if there exists a prerealisable polynomial F' satisfying $N(F') = G(F)$, $N(F')$ must be a polygon since there exists a vertex of degree 2. Then every vertex of $N(F')$ must be of degree 2, whereas $G(F)$ has a vertex of degree 3.

Now we explain an important feature of adjacency. Let $\{1, 4, 6\}$ be the current critical path. Suppose we decrease the time cost of the activity 1. Then the critical path may change. The new one should be $\{2, 3, 6\}$, because that is the only path which is adjacent to $\{1, 4, 6\}$ and does not contain 1 as seen in the adjacency graph. Or suppose the time cost of the activity 5 has suddenly risen up. Then $\{1, 4, 5\}$, not $\{2, 5\}$, is the only possibility.

One should study $V(F)$ to obtain $G(F)$, which is not very easy in general. When the PERT chart has some structure, one might obtain $G(F)$ accordingly. We shall discuss three cases: independency, product structure and homogeneity.

First, an independent activity increases the connectivity of the adjacency graph.

Proposition 7. *If a term t_I of F contains a variable t_i which does not appear in any other terms, then the vertex I of $G(F)$ is connected to all other vertices.*

Proof. Let t_J be another term of F. $\boldsymbol{e}_J$ is an interior point of C_J. Take the point $\boldsymbol{t} = \boldsymbol{e}_J + s\boldsymbol{e}_i$ for $s \geq 0$. When s increases from 0, t_I is the only term in F whose value at $\boldsymbol{t}$ gets larger. Eventually we get $t_I = t_J$. Those t_I, t_J are the only terms that attain the maximum at that $\boldsymbol{t}$. Thus C_I and C_J are adjacent. □

Next we treat the product. For two undirected graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, we define the Cartesian product $G_1 \square G_2$ as in [2], that is:

1. the vertex set being $V_1 \times V_2$,
2. $\{(v_1, v_2),\ (v_1', v_2')\}$ is an edge of $G_1 \square G_2$ iff

$$\{v_1, v_1'\} \in E_1,\ v_2 = v_2' \quad \text{or} \quad v_1 = v_1',\ \{v_2, v_2'\} \in E_2.$$

Proposition 8. *Let F_1, F_2 be polynomials with no common variables. Then $G(F_1F_2) = G(F_1) \square G(F_2)$.*

Proof. Let the number of variables of F_1, F_2 be n, m, respectively. Since $V(F_1F_2)_{\geq 0} = \big(V(F_1)_{\geq 0} \times \boldsymbol{R}^m_{\geq 0}\big) \cup \big(\boldsymbol{R}^n_{\geq 0} \times V(F_2)_{\geq 0}\big)$ holds, we have the proposition. □

Remark 3. If F_1, F_2 are realisable, then one can also realise F_1F_2 by connecting the ending node of the PERT chart of F_1 to the starting node of the PERT chart of F_2.

The Cartesian product of n copies of K_2 is called the *n-hypercube graph* and denoted by Q_n (see [2]).

Corollary 2. *Q_n is the adjacency graph of the realisable tropical polynomial $F = \prod_{k=1}^{n} (t_{i_k} \oplus t_{j_k})$.*

Proof. By Example 1, K_2 is the adjacency graph of $F_{2,1}$. Therefore, $Q_n = (K_2)^{\square n}$ is that of the n product of $F_{2,1}$ with independent variables. □

Finally we treat the homogeneity.

Proposition 9. *If F is homogeneous, then $G(F) = N(F)$.*

Proof. Put $\deg F = d$, and fix a point $\boldsymbol{t}$. Then the value of each term of F at $\boldsymbol{t} + (a, \ldots, a)$ is exactly $a \cdot d$ bigger than the value at $\boldsymbol{t}$, so if two chambers are adjacent, they are also adjacent within Γ. □

For a prerealisable tropical polynomial $F = \sum_{I \in \mathcal{I}} t_I$, put $F^\vee = \sum_{I \in \mathcal{I}} t_{[n]\setminus I}$.

Lemma 1. *$F^\vee$ is prerealisable.*

Proof. If $[n] \setminus I \subset [n] \setminus J$ holds, $I \supset J$ also holds and the nondivisibility condition of F yields $I = J$. Thus we have $[n] \setminus I = [n] \setminus J$. □

Remark 4. $F^\vee$ may not be realisable even if F is realisable. For instance, Corollary 1 show that $F_{3,1}$ is realisable, whereas $F^\vee_{3,1} = F_{3,2}$ is not.

Proposition 10. *If F is homogeneous, then $G(F) = G(F^\vee)$.*

Proof. For a point p, denote the point symmetric about $\mathbf{0} = (0, \ldots, 0)$ as $-p$. Also denote the function taking the inverse value of a tropical monomial function t_I with respect to the tropical multiplication (usual addition) as t_I^{-1}.

Since t_I defines a linear function passing through $\mathbf{0}$, its graph is symmetric about $\mathbf{0}$. Thus the value of t_I^{-1} at $-p$ and the value of t_I at p are the same.

Tropically adding these monomials, we obtain $F = \sum_I t_I$ and $F' = \sum_I t_I^{-1}$. The value of F at p coincides to the value of F' at $-p$, so the corner locus of F and the corner locus of F' are symmetric about $\mathbf{0}$. By tropically multiplying $t_{[n]}$ to F', we have $F^\vee$ because $t_I^{-1} \otimes t_{[n]} = t_{[n]\setminus I}$. Since tropically multiplying a monomial does not change the corner locus, $V(F)$ and $V(F^\vee)$ are symmetric about $\mathbf{0}$. Then we have $N(F) = N(F^\vee)$ because of the duality between the Newton subdivision and the tropical hypersurface, and the proposition follows from Proposition 9. □

Since $F^\vee_{n,k} = F_{n,n-k}$ holds, we have the following corollary.

Corollary 3. *$G(F_{n,k})$ and $G(F_{n,n-k})$ coincides.*

Example 4. From the previous corollary and Example 1, we obtain

$$G(F_{n,n-1}) = G(F_{n,1}) = K_n.$$

Acknowledgements

We thank to Stéphane Gaubert and Jean-Jacques Risler for discussion and kindly noticing us good references. M. K. is supported by Grant-in-Aid for Scientific Research 21540045. S. O. thanks the support by the MEXT program "Support Program for Improving Graduate School Education".

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