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A transformation formula for a certain Eisenstein series in aerodynamic interference calculations

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Abstract. In 1949, F. Olver established a transformation formula which converts a certain slowly convergent series into a rapidly convergent and easily computable form. The original (double) series occurs in aerodynamic interference calculations, and its numerical estimates have some practical importance. In this paper, the author revisits this double series from the point of view of analytic number theory, and shows the transformation property as a corollary of the Fourier-type expansion of a certain kind of non-holomorphic Eisenstein series by employing Mellin-Barnes integrals.

Keywords. Eisenstein series, aerodynamic interference calculations, Mellin-Barnes formula

1. Introduction

The principal purpose of this paper is to introduce a certain kind of Eisenstein series and its transformation formula which is relevant to aerodynamic interference calculations.

In the airplane aerodynamic development, wind tunnel tests play an important role as one of the main tools of computational and experimental fluid dynamics. Historically, the rigorous systematic mathematical analyses for the science of aerodynamics was initiated by Ludwig Prandtl (1875–1953) in Göttingen, who investigated the effect on an aerofoil in a wind tunnel of a circular cross-section. Glauert developed and summarised their pioneering works in [5], where evaluating wall interference methods are served. Their mathematical models have progressed enough accurate for the estimation under suitable conditions, nevertheless, further developments, for example new understanding of wind tunnel wall influence and advanced numerical fluid dynamics codes, are expected (see [4]).

In this paper, we revisit double series (see (2.1), (3.1)) occur in aerodynamic interference calculations (cf. [5] and [1, 2]). At the request of aerodynamicists, F. Olver [7] established one transformation formula which converts the original slowly convergent series into a rapidly convergent and easily computable form (Theorem in Sec. 3). His result can be regarded as a Fourier-type expansion of a kind of Eisenstein series from the perspective of the analytic number theory. We give comprehensive definitions for these double series (see (2.2), (3.2)). Adopting the Mellin-Barnes integral transform technique developed precisely by M. Katsurada [6] in the study of the asymptotic expansion of the Epstein zeta-function, it is possible to obtain the transformation formula for our Eisenstein series (Theorem 1 in Sec. 3) which provides a new proof of Olver's Theorem.

2. Classical results

Throughout this paper aerofoils are assumed to be in rectangular channels. Let s be the semi-span of the aerofoil and $\mu = h/b$ be the ratio of tunnel height h to tunnel breadth b. According to H. Glauert [5, Chap. 14.4], we introduce one example of the double series and its converted form in the effect of the tunnel walls to cause an upward inclination of the stream:

$$\begin{aligned}\varepsilon_0 &= \frac{s}{\pi b}\cdot\sum_{n=-\infty}^{\infty}{}'\sum_{m=-\infty}^{\infty}(-1)^n\frac{m^2-\mu^2n^2}{(m^2+\mu^2n^2)^2}\\ &= \frac{s}{\pi b}\cdot\left\{\frac{\pi^2}{3}+8\pi^2\sum_{n=1}^{\infty}\frac{n}{1+\exp(2\mu\pi n)}\right\},\end{aligned}\tag{2.1}$$

where $\sum_n'\sum_m$ denotes the sum of the pair of integers (n,m) excluding $(0,0)$. In [5], the notation $\epsilon_1 = \varepsilon_0\cdot C_L\mathrm{S}h/4s\mathrm{C}$ is employed. Here C_L is the lift coefficient, S is the area of the aerofoil and $\mathrm{C} = hb$ is the area of tunnel section. The notation ε_0 corresponds to f_0 in Rosenhead's exact formula for the induced velocity (cf. [9]).

The second equality in (2.1) was deduced by using the partial fraction decomposition of $\cot z$ in [5]. From the point of view of the number theory, the double series ε_0 is one of the holomorphic Eisenstein series at the specific point on the upper half plane, and the second expression is regarded as a kind of Lambert series. In the following of this section, we rewrite the above instance in the words of arithmetic functions. Let $\mathcal{H} = \{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}$ be the complex upper half plane. For $z\in\mathcal{H}$, we define a certain kind of the holomorphic Eisenstein series by

$$\mathcal{T}_k(z) = \sum_{n=-\infty}^{\infty}{}'\left\{\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{(m+nz)^k}\right\},\tag{2.2}$$

for even $k \geqq 2$. Then, it is readily ascertained that the Fourier series expansion

$$\mathcal{T}_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{l=1}^{\infty}\widehat{\sigma_{k-1}}(l)\exp(2\pi ilz)$$
$$= 2\zeta(k) - \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\frac{n^{k-1}}{1+\exp(-2\pi inz)}$$

holds. Here $\zeta(s)$ is the Riemann zeta-function and $\widehat{\sigma_{k-1}}(l) = \sum_{n|l}(-1)^{l/n}n^{k-1}$. As a consequence, the transformation of ε_0 is verified again by the following identity:

$$\varepsilon_0 = \frac{s}{\pi b}\cdot\mathcal{T}_2(\mu\sqrt{-1}).$$

3. Main Theorem

Let $(\alpha,\beta) \in \mathbb{R}^2$ and μ be a positive parameter. In the study of the upwash angle at the tail due to tunnel interference, the double series $F(\alpha,\beta)$ defined as below was approximately evaluated by L. W. Bryant and H. C. Garner (cf. [2], see also [1]):

$$F(\alpha,\beta) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^n[f_n(\alpha-m) - f_n(\beta-m)], \quad (3.1)$$

where

$$f_n(y) = \frac{y(y^2+2\mu^2n^2)}{n^2(y^2+\mu^2n^2)^{3/2}}.$$

Here, y-axis is along the span of the aerofoil. (In the background, x-axis extends forward in the flow direction and z-axis indicates the downwash direction.) In [7], Olver proved the following theorem after the collaboration of Garner who supplied him the aerodynamic information.

Theorem (F. Olver 1949)**.** *The double series $F(\alpha,\beta)$ attached to aerodynamic calculation is able to transformed as follows:*

$$F(\alpha,\beta) = \chi(\beta) - \chi(\alpha),$$

where

$$\chi(u) = \frac{1}{6}\pi^2 u + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(2\pi mu)\Big\{8\pi\mu^2 m(-1)^{n-1} \times K_0(2\pi\mu mn) + 4\mu\frac{(-1)^{n-1}}{n}K_1(2\pi\mu mn)\Big\},$$

and $K_\nu(z)$ denotes the modified Bessel function.

Remark 1. Olver proved his theorem starting from the contour integral expression comes from the residue theorem such that

$$\sum_{m=-\infty}^{\infty}[f_n(\alpha-m) - f_n(\beta-m)] = \int_C [f_n(\alpha-\xi) - f_n(\beta-\xi)]\frac{\cot\pi\xi}{2i}d\xi,$$

for some contour C (see [7] and also [8, Chap. 8 §7]). On the note for the preceding paper, G. Reuter (1949) pointed out the Theorem above can be proved by using the Poisson summation formula.

Our main purpose is to interpret Olver's result in terms of some arithmetic functions, and deduce the transformation formula via the Mellin-Barnes type integrals which gives a new proof of Olver's theorem.

Definition 1. We define a certain kind of the non-holomorphic Eisenstein series (or the Epstein zeta-function) attached to aerodynamic interference calculations by

$$\mathcal{T}(u;s;z) = \sum_{n\in\mathbb{Z}\setminus\{0\}}\left\{\lim_{M\to\infty}\sum_{m=-M}^{M}f_{m,n}(u;s;z)\right\}, \quad (3.2)$$

for $z = x+\sqrt{-1}\,y \in \mathcal{H}$, real parameter $u \in \mathbb{R}\setminus\mathbb{Z}$ and complex parameter $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here we choose

$$f_{m,n}(u;s;z) = \frac{(-1)^n}{n^2}\cdot\frac{\mathrm{Re}(m-u+nz)}{|(m-u)+nz|}\left\{1+\frac{n^2|z|^2}{|(m-u)+nz|^s}\right\}.$$

Then the main result is described as follows:

Theorem 1. *Let $z = x+\sqrt{-1}\,y \in \mathcal{H}$, $\mathrm{Re}(s) > 1$ for $s \in \mathbb{C}$ and put $u_n = u - nx$. Assume $u_n \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$. Then the transformation formula*

$$\mathcal{T}(u;s;z) = \frac{\pi^2}{3}u + 4y\sum_{n\neq 0}\sum_{m=1}^{\infty}(-1)^{n-1}|n|^{-1} \times \sin(2\pi mu_n)K_1(2Y) + \frac{8\pi^s|z|^2}{\sqrt{\pi}(s-1)\Gamma(\frac{s}{2}-\frac{1}{2})}\sum_{n\neq 0}\sum_{m=1}^{\infty}(-1)^{n-1} \times m^{s-1}\cdot\sin(2\pi mu_n)Y^{1-\frac{s}{2}}K_{\frac{s}{2}-1}(2Y)$$

holds. Here the notation $Y = \pi ym|n|$ is used. The last two double series on the right side converges absolutely for all complex s, and hence the formula provides the meromorphic continuation of $\mathcal{T}(u;s;z)$ to the whole s-plane $\mathbb{C}$.

Remark 2. The double series $\mathcal{T}(u;s;z)$ is not a generalization of $\mathcal{T}_k(z)$, and hence the transformation formula of $\mathcal{T}_k(z)$ does not follow from Theorem 1. The relation $\mathcal{T}_k(z)$ and $\mathcal{T}(u;s;z)$ can be compared to the relation of the holomorphic and the non-holomorphic Eisenstein series attached to $\mathrm{SL}(2,\mathbb{Z})$.

4. Proof of the main theorem

First, we introduce the Mellin-Barnes formula for binomial functions. For the convenience, we frequently use the notation

$$\Gamma\begin{pmatrix}a_1,a_2,\dots,a_k\\ b_1,b_2,\dots,b_l\end{pmatrix} = \frac{\prod_{i=1}^{k}\Gamma(a_i)}{\prod_{j=1}^{l}\Gamma(b_j)}.$$

Lemma 1. *For $a \in \mathbb{C}$ with $a \notin \mathbb{Z}_{\le 0}$ and $Z \in \mathbb{C}$, the formula*

$$(1+Z)^{-a} = \frac{1}{2\pi i}\int_C \Gamma\binom{-w, a+w}{a} Z^w\, dw,$$

holds, where $|\arg(Z)| \le \pi - \delta$ with any small $\delta > 0$, and the path integration is denoted by C taken from $-i\infty$ to $i\infty$ so as to separate the poles of $\Gamma(a+w)$ and $\Gamma(-w)$.

Proof. This Lemma is indicated in [10, 14.51] as a corollary of Barnes' contour integral of Gauss' hypergeometric function. □

Next, for $w \in \mathbb{C}$, we let

$$\zeta_X(w) = \sum_{m=1}^{\infty} \exp(2\pi i m X) m^{-w} \quad (\mathrm{Re}(w) > 1)$$

be the exponential zeta-function, which is continued to an entire function over the w-plane if $X \in \mathbb{R} \setminus \mathbb{Z}$. In the proof of Theorem 1, the functional equation of a kind of the *bilateral* Hurwitz zeta-function will be applied.

Lemma 2. *Let $X \in \mathbb{R} \setminus \mathbb{Z}$, $w \in \mathbb{C}$. Then the functional equation*

$$\sum_{m=-\infty}^{\infty} (m-X)|m-X|^{2w} = \frac{i\Gamma(w+\frac{3}{2})}{\pi^{2w+\frac{3}{2}}\Gamma(-w)}\left\{\zeta_X(2w+2) - \zeta_{-X}(2w+2)\right\}$$

holds for $\mathrm{Re}(w) < -1$*. The formula provides the meromorphic continuation of the function defined on the left hand side to the whole w-plane $\mathbb{C}$.*

Proof. A slight modification of the argument given in [6, Sect. 3 Lemma 1] with the duplication and reflection formulas of the Gamma function

$$\Gamma(2w+2) = \frac{2^{(2w+2)-1}}{\sqrt{\pi}}\Gamma(w+1)\Gamma(w+\tfrac{3}{2}),$$
$$\sin(\pi w) = \frac{\pi}{\Gamma(1-w)\Gamma(w)} = \frac{-\pi}{w\Gamma(-w)\Gamma(w)},$$

gives the proof of Lemma 2. □

Proof of Theorem 1. First, let $u(m) = m - u \in \mathbb{R}$, $s_0 \in \mathbb{C}$, and write

$$\frac{\mathrm{Re}(u(m)+nz)}{|u(m)+nz|^{s_0}} = \frac{\mathrm{Re}(u(m)+nz)}{|ny|^{s_0}}\left(1+\left|\frac{u(m)+nx}{ny}\right|^2\right)^{-\frac{s_0}{2}}.$$

By taking $Z = |(u(m)+nx)/ny|^2$ in the Mellin-Barnes formula in Lemma 1, the expression

$$\left(1+\left|\frac{u(m)+nx}{ny}\right|^2\right)^{-s_0/2} = \frac{1}{2\pi i}\int_{(c)} \Gamma\begin{pmatrix}-w, \frac{s_0}{2}+w \\ \frac{s_0}{2}\end{pmatrix}\left|\frac{u(m)+nx}{ny}\right|^{2w} dw$$

follows for any $n \in \mathbb{Z} \setminus \{0\}$. Here (c) denotes the vertical straight path from $c - i\infty$ to $c + i\infty$, and $\mathrm{Re}(s_0) > 0$ and $-\mathrm{Re}(s_0)/2 < c < 0$ are temporarily assumed. We substitute the expression above into each term of the defining inner series of $\mathcal{T}(u;s;z)$ in (3.2), and shift the paths of the integrals from (c) to (c') with $-(1+\mathrm{Re}(s))/2 < c' = c-1 < -1$ to obtain

$$\begin{aligned}
&\lim_{M\to\infty}\sum_{m=-M}^{M} f_{m,n}(u;s;z)\cdot(-1)^{-n}n^2 \\
&= \lim_{M\to\infty}\sum_{m=-M}^{M}\left\{\frac{1}{2\pi i}\int_{(c')}\Gamma\begin{pmatrix}-w, \frac{1}{2}+w \\ \frac{1}{2}\end{pmatrix}\left|\frac{u(m)+nx}{ny}\right|^{2w} dw \right.\\
&\quad\left. + \frac{n^2|z|^2}{2\pi i}\int_{(c')}\Gamma\begin{pmatrix}-w, \frac{1}{2}+\frac{s}{2}+w \\ \frac{1}{2}+\frac{s}{2}\end{pmatrix}\frac{|u(m)+nx|^{2w}}{|ny|^{2w+s}} dw\right\} \\
&\quad\times\frac{\mathrm{Re}(u(m)+nz)}{|ny|} + \lim_{M\to\infty}\sum_{m=-M}^{M}\mathrm{sign}\,(u(m)+nx),
\end{aligned} \tag{4.1}$$

where the last sum arises from the residues of the simple poles of $\Gamma(1/2+w)$ at $w = -1/2$. Under the assumptions $-(1+\mathrm{Re}(s))/2 < c' < -1$ and $\mathrm{Re}(s) > 1$, the interchange of the integrals and summations is justified by the absolute convergence of the resulting integrals. Applying Lemma 2 to the infinite sum by taking $X = u_n = u - nx \in \mathbb{R} \setminus \mathbb{Z}$ to find that (4.1) is equal to

$$\begin{aligned}
&\frac{1}{2\pi i|ny|}\int_{(c')}\frac{\pi^{-\frac{3}{2}}}{2(\pi|ny|)^{2w}}\Gamma\begin{pmatrix}\frac{3}{2}+w, \frac{1}{2}+w \\ \frac{1}{2}\end{pmatrix} \\
&\quad\times 2i\left\{\zeta_{u_n}(2w+2) - \zeta_{-u_n}(2w+2)\right\} dw \\
&+ \frac{n^2|z|^2}{2\pi i|ny|^{s+1}}\int_{(c')}\frac{\pi^{-\frac{3}{2}}}{2(\pi|ny|)^{2w}}\Gamma\begin{pmatrix}\frac{3}{2}+w, \frac{1}{2}+\frac{s}{2}+w \\ \frac{1}{2}+\frac{s}{2}\end{pmatrix} \\
&\quad\times 2i\left\{\zeta_{u_n}(2w+2) - \zeta_{-u_n}(2w+2)\right\} dw \\
&\quad+ \lim_{M\to\infty}\sum_{m=-M}^{M}\mathrm{sign}\,(m-u_n).
\end{aligned} \tag{4.2}$$

Shifting back the path of the first integral from (c') to (c), and with the use of the Fourier series of the sawtooth wave

$$-2u_n - \lim_{M\to\infty}\sum_{m=-M}^{M}\mathrm{sign}\,(m-u_n) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi m u_n)}{m},$$

we confirm that the residues, coming from the simple poles of $\Gamma(1/2+w)$ at $w = -1/2$, almost cancel with the third terms in (4.2). Changing the orders of integrals and summations in (4.2) again, and changing the variables w as $v - 1$ for the second integral in (4.2), we observe that (4.2) is equal to

$$
\begin{aligned}
&-\frac{\pi^{-\frac{3}{2}}}{|ny|}\sum_{m=1}^{\infty} m^{-2}\sin(2\pi m u_n)\\
&\times\frac{1}{2\pi i}\int_{(c)}\Gamma\begin{pmatrix}\frac{3}{2}+w,\frac{1}{2}+w\\ \frac{3}{2}\end{pmatrix}(\pi y m|n|)^{-2w}dw\\
&-\frac{4\pi^{-\frac{3}{2}}n^2|z|^2}{|ny|^{s+1}(s-1)}\sum_{m=1}^{\infty} m^{-2}\sin(2\pi m u_n)(\pi y m|n|)^2\\
&\times\frac{1}{2\pi i}\int_{(c)}\Gamma\begin{pmatrix}\frac{s}{2}-\frac{1}{2}+v,\frac{1}{2}+v\\ \frac{s}{2}-\frac{1}{2}\end{pmatrix}(\pi y m|n|)^{-2v}dv\\
&-2u_n. \qquad (4.3)
\end{aligned}
$$

We have used $\Gamma(1/2) = 2\Gamma(3/2)$ (resp. $\Gamma(\frac{1}{2}+\frac{s}{2}) = (\frac{s}{2}-\frac{1}{2})\Gamma(\frac{s}{2}-\frac{1}{2})$) in the first (resp. second) integral in (4.3). Employing the Barnes integral representation of the confluent hypergeometric function due to Katsurada ([6, Lemma 3]) such that

$$U(s;2s;4Z) = \frac{e^{2Z}\Gamma(s)}{\sqrt{\pi}(2Z)^{2s}}\cdot\frac{1}{2\pi i}\int_{(c)}\Gamma\begin{pmatrix}s+w,\frac{1}{2}+w\\ s\end{pmatrix}Z^{-2w}dw,$$

for $\mathrm{Re}(s) > 0$ and $Z > 0$ with the fact that

$$U(s;2s;4Z) = \frac{2Z^{s+1/2}e^{2Z}}{\sqrt{\pi}(2Z)^{2s}}K_{s-\frac{1}{2}}(2Z)$$

(cf. [3, 6.9.1.(13)]), we obtain the equality in Theorem 1. Here $U(s_1;s_2;z)$ is the confluent hypergeometric function of the second kind defined by

$$U(s_1;s_2;Z) = \frac{1}{\Gamma(s_1)}\int_0^{\infty} e^{-Zu}u^{s_1-1}(1+u)^{s_2-s_1-1}du,$$

for $\mathrm{Re}(s_1) > 0$ and $|\arg(Z)| < \pi/2$ (cf. [3, 6.5.(3)]). The analytic continuation is established by using

$$K_\nu(Z) \sim (\pi/2Z)^{\frac{1}{2}}e^{-Z}, \qquad (Z \to +\infty)$$

for $\nu \in \mathbb{C}$ and $|\arg(Z)| < \pi/2$ (cf. [3, 7.4.1.(1)]). □

Remark 3. Specifying $z = \sqrt{-1}\mu \in \mathcal{H}$ and $s = 2$ in Theorem 1, and replacing the n-sum so as $\sum_{n=1}^{\infty}$, we obtain Olver's transformation formula as a corollary.

For further discussion, first we note that our main contribution is to provide extensive understanding of some double series occurs in aerodynamic interference calculations by introducing a kind of non-holomorphic Eisenstein series. However, the fact $\mathcal{T}(u;s;z)$ does not satisfy an ordinary modular transformation formula, presents a future consideration. In the next step, if we find some double series which satisfies modular transform in the wind tunnel interference, the modular relation suggests that there exists some scale changing of the wind tunnel. Furthermore, the author expects that a functional equation (if it exists) will imply more deep results in the theory of aerodynamic interference.

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References

[1] W. S. Brown.: Wind tunnel corrections on ground effect, *Aeronautical Research Council Reports and Memoranda* **1865** (1938).

[2] L. W. Bryant and H. C. Garner.: Control testing in wind tunnels, *Aeronautical Research Council Reports and Memoranda* **2881** (1951).

[3] A. Erdélyi, et al.: Higher transcendental functions, Vol. I, II, McGraw-Hill, New York, 1953.

[4] B. F. R. Ewald, et al.: Wind Tunnel Wall Correction, AGARDograph **336** published by the North Atlantic Treaty Organization, 1998.

[5] H. Glauert.: The Elements of Aerofoil and Airscrew Theory, Cambridge University Press, 1926.

[6] M. Katsurada.: Complete asymptotic expansions associated with Epstein zeta-functions, *Ramanujan J.* **14** (2007) 249–275.

[7] F. W. J. Olver.: Transformation of certain series occurring in aerodynamic interference calculations, *Quart. J. Mech. Appl. Math.* **2** (1949) 452–457.

[8] F. W. J. Olver.: Asymptotics and Special Functions, Academic Press, 1974. A K Peters, 1997.

[9] L. Rosenhead.: The effect of wind tunnel interference on the characteristics of an aerofoil, *Proc. R. Soc. Lond. A* **129** (1930) 135–145.

[10] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, (4th ed). Cambridge University Press, Cambridge, 1927.

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