Non-convex anisotropic surface energy and zero mean curvature surfaces
in the Lorentz-Minkowski space

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Abstract. An anisotropic surface energy functional is the integral of an energy density function over a surface. The energy density depends on the surface normal at each point. The usual area functional is a special case of such a functional. We study stationary surfaces of anisotropic surface energies in the euclidean three-space which are called anisotropic minimal surfaces. For any axisymmetric anisotropic surface energy, we show that, a surface is both a minimal surface and an anisotropic minimal surface if and only if it is a right helicoid. We also construct new examples of anisotropic minimal surfaces, which include zero mean curvature surfaces in the three-dimensional Lorentz-Minkowski space as special cases.

Keywords. anisotropic, mean curvature, minimal surface, zero mean curvature surface, Lorentz-Minkowski space, Wulff shape

1. Introduction

Let \( \gamma : \Omega \to \mathbb{R}_+ \) be a positive \( C^\infty \) function on a nonempty open set \( \Omega \) of the two-dimensional unit sphere \( S^2 := \{ X \in \mathbb{R}^3 : |X| = 1 \} \). Let \( X : \Sigma \to \mathbb{R}^3 \) be an immersion from a two-dimensional oriented connected compact \( C^\infty \) manifold \( \Sigma \) (with or without boundary) to the three-dimensional euclidean space \( \mathbb{R}^3 \). Denote by \( \nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2 \) the unit normal along \( X \) (in other words, the Gauss map of \( X \)). If \( \nu(\Sigma) \subseteq \Omega \), we say that \( X \) is compatible with \( \gamma \) and we define the following functional.

\[
\mathcal{F}[X] = \int_{\Sigma} \gamma(\nu) \, d\Sigma,
\]

where \( d\Sigma \) is the area element of \( X \). Such a functional is used to model anisotropic surface energies. Applications can be found in many branches of the physical sciences including metallurgy and crystallography ([14, 15]). We will call \( \mathcal{F}[X] \) the anisotropic energy of \( X \), and \( \gamma \) the energy density function.

We call stationary surfaces of (1) for compactly-supported variations \( \gamma \)-minimal surfaces. It is obvious that, for \( \gamma \equiv 1 \), \( \gamma \)-minimal surfaces are usual minimal surfaces.

Denote by \( D\gamma \) and \( D^2\gamma \) the gradient and the Hessian of \( \gamma \) on \( \Omega \), respectively. Denote by \( \nu \) the identity endomorphism field on the tangent space \( T_{\nu}(S^2) \). If the matrix \( D^2\gamma + \gamma I \) is non-singular at each point \( \nu \) in \( \Omega \), a mapping \( \gamma : \Omega \to \mathbb{R}^3 \) defined by \( Y(\nu) = D\gamma + \gamma(\nu)\nu \) is an immersion and \( Y \) defines the uniquely determined immersed surface with unit normal \( \nu \) whose support function coincides with \( \gamma \), that is \( \gamma(\nu) = \langle Y(\nu), \nu \rangle \) holds. We say that \( Y \) is the standard body for \( \gamma \). (As for the terminology “standard body”, we quote [12,].) We will sometimes use the symbol \( M_\gamma \) to represent the mapping \( Y \) or the image \( Y(\Omega) \) of \( Y \).

We say that \( \gamma : \Omega \to \mathbb{R}_+ \) satisfies the convexity condition, if the matrix \( D^2\gamma + \gamma I \) is positive definite at each point \( \nu \) in \( \Omega \). In this case, the standard body \( M_\gamma \) for \( \gamma \) is strongly convex (that is, the principal curvatures of \( M_\gamma \) are positive everywhere), and the functional \( \mathcal{F} \) appearing in (1) is called a constant coefficient parametric elliptic functional, and stationary surfaces are extensively studied in recent years.

In this paper, we do not assume the convexity condition. By this generalization, we obtain a more variety of important examples. For example, zero mean curvature immersions in the Lorentz-Minkowski space \( \mathbb{R}^3_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : dx^2 = dx_1^2 + dx_2^2 - dx_3^2 \} \) arise as \( \gamma \)-minimal surfaces for a certain simple function \( \gamma \) as follows (cf. §3).

Theorem 1. Set \( \Omega_1 := \{ \nu = (\nu_1, \nu_2, \nu_3) \in S^2 : |\nu_3| > \sqrt{2}/2 \} \), \( \Omega_2 := \{ \nu \in S^2 : |\nu_3| < \sqrt{2}/2 \} \). Define a function \( \gamma : S^2 \to \mathbb{R} \) as \( \gamma(\nu) = \sqrt{|\nu_3^2 - \nu_1^2 - \nu_2^2|} = \sqrt{2|\nu_3^2| - 1} \). Then, an immersion \( X : \Sigma \to \mathbb{R}^3 \) with Gauss image \( \nu(\Sigma) \subseteq \Omega_1 \cup \Omega_2 \) is \( \gamma \)-minimal if and only if the mean curvature of \( X \) is zero as an immersed surface in \( \mathbb{R}^3_1 \).

This result indicates that the recent investigations about zero mean curvature surfaces in \( \mathbb{R}^3_1 \) changing their causal type across null curves (regular curves whose velocity vector fields are lightlike) or lightlike lines from spacelike zero mean curvature surfaces to timelike zero mean curvature surfaces ([3, 6, 5, 4, 2]) should be very natural and reasonable. Probably the most well-known example of such surfaces is the right helicoid with the timelike axis as its axis, which changes its causal type across a null curve from a spacelike zero mean curvature surface to a timelike zero mean curvature surface ([3, 6]). In §4, we will show a more
general remarkable result as follows.

**Theorem 2.** Let $\gamma: \Omega \to \mathbb{R}^+_1$ be a positive $C^\infty$ function on a nonempty open set $\Omega$ in $S^2$. Assume that the matrix $D^2\gamma + \gamma I$ is non-singular at each point $\nu \in \Omega$. Assume also that $\gamma$ is axisymmetric and not a constant function.

Let $X: \Sigma \to \mathbb{R}^3$ be an immersion which is compatible with $\gamma$. Then, $X$ is both minimal and $\gamma$-minimal if and only if it is a part of either a plane or a right helicoid whose axis is parallel to the axis of $\gamma$.

This result is a generalization of [7, Theorem 4.2] and a refinement of [9, Proposition III.1]. [7, Theorem 4.2] proves that a spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the timelike axis are only both a minimal surface in the euclidean space $\mathbb{R}^3$ and a spacelike zero mean curvature surface in $\mathbb{R}^3$. [9, Proposition III.1] proves that a right helicoid is a $\gamma$-minimal surface for any axisymmetric $\gamma$ whose axis is parallel to the axis of the helicoid itself.

Theorem 2 combined with Theorem 1 implies the following:

**Corollary 1.** A spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the $x_3$-axis are only both a minimal surface in the euclidean space $\mathbb{R}^3$ and a spacelike zero mean curvature surface in $\mathbb{R}^3$. Also, a timelike plane and the timelike part of a right helicoid whose axis is parallel to the $x_3$-axis are only both a minimal surface in $\mathbb{R}^3$ and a timelike zero mean curvature surface in $\mathbb{R}^3$.

In general, it is not easy to construct examples of $\gamma$-minimal surfaces. For any axisymmetric energy density function $\gamma$, there exist $\gamma$-minimal surfaces which are also symmetric with respect to the same axis as $\gamma$. The existence theorem and a certain kind of representation formula of these surfaces were given in [8] and they were called anisotropic catenoid. Although the convexity condition for $\gamma$ was assumed in [8], the method there works also for non convex $\gamma$. In this paper, for certain classes of $\gamma$, we will give another type of examples of $\gamma$-minimal surfaces which are foliated by parallel circles but are not surfaces of revolution. We will call them $\gamma$-minimal surfaces of Riemann-type after Riemann's minimal surfaces in $\mathbb{R}^3$.

**Proposition 1.** Let $\gamma: \Omega \to \mathbb{R}^+_1$ be a positive $C^\infty$ function on a nonempty open set $\Omega$ in $S^2$. Assume that the matrix $D^2\gamma + \gamma I$ is non-singular at each point $\nu \in \Omega$. We also assume that the standard body $M_\gamma$ for $\gamma$ is a qudratic surface of revolution. Then, there are $\gamma$-minimal surfaces of Riemann-type.

From Theorem 1, we see that spacelike and timelike zero mean curvature surfaces of Riemann-type in $\mathbb{R}^3$ are obtained as special cases of surfaces given by Proposition 1. Actually, for $\gamma|_{\Omega_1}$ in Theorem 1, $M_\gamma$ is a hyperboloid of two sheets, and for $\gamma|_{\Omega_2}$, $M_\gamma$ is a hyperboloid of one sheet ($\S 5$, Lemma 5).

We should remark that zero mean curvature surfaces of Riemann-type in $\mathbb{R}^3$ were studied also in [10, 11].

In $\S 5$, for $\gamma$ satisfying the assumption in Proposition 1, we will give explicit parameter representations of all $\gamma$-minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the rotation axis of $M_\gamma$ (Proposition 3). Actually, Proposition 1 is a corollary of Proposition 3.

Some of the results in this article can be generalized to hypersurfaces in $\mathbb{R}^{n+1}$.

## 2. Preliminaries

In this section, we give the definitions of the Wulff shape, anisotropic mean curvature, and their fundamental properties and representation formulas. We quote [12, 1, 8] as references.

Let $\gamma: \Omega \to \mathbb{R}^+$ be a positive $C^\infty$ function on a nonempty open set $\Omega$ of the unit sphere $S^2$. Assume that the matrix $D^2\gamma + \gamma I$ is non-singular at each point $\nu \in \Omega$.

If $\Omega = S^2$, then, for any $V > 0$, there exists a uniquely determined (up to translations in $\mathbb{R}^3$) convex surface $W(V)$ such that $W(V)$ attains the minimum of $\mathcal{F}$ among all closed piecewise smooth surfaces in $\mathbb{R}^3$ enclosing the $3$-dimensional volume $V$ ([13]). For the special value $V_0 = (1/3) \int_{S^2} \gamma(\nu) \, dS^2$, $W(V_0)$ is called the Wulff shape for $\gamma$, and we will denote it by $W$. In the special case where $\gamma \equiv 1$, $F[X]$ is the usual area of the surface $X$ and $W$ is the unit sphere $S^2$. In general, $W$ is not smooth. $W$ is a smooth strongly convex surface if and only if $\gamma$ satisfies the convexity condition (see (1)). In this case, $W$ can be parametrized by the smooth mapping

$$Y: S^2 \to \mathbb{R}^3, \quad Y(\nu) = D\gamma + \gamma(\nu)\nu,$$

where we regard $D\gamma$ at $\nu \in S^2$ as a point in $\mathbb{R}^3$ in the canonical manner. We remark that the outward unit normal to $W$ at point $Y(\nu)$ coincides with $\nu$. And the function $\gamma$ coincides with the support function of $W$, that is $\gamma(\nu) = \langle Y(\nu), \nu \rangle$, where $\langle , \rangle$ is the inner product in $\mathbb{R}^3$. This means that $W$ is the standard body for $\gamma$.

Let $X: \Sigma \to \mathbb{R}^3$ be an immersion. By parallel translation in $\mathbb{R}^3$, $D\gamma$ may be considered as a smooth tangent vector field along $X$. Let $X_\gamma = X + \epsilon \partial X + O(\epsilon^2)$ be a smooth, compactly supported variation of $X$. The anisotropic mean curvature $\Lambda$ of $X$ is defined by the first variation formula ([8])

$$\delta \mathcal{F} := \partial_s \mathcal{F}[X_\epsilon]_{s=0} = - \int_{\Sigma} \Lambda(\delta X, \nu) \, d\Sigma, \quad (2)$$

$$\Lambda := -\text{trace}_{\Sigma}(D^2\gamma + \gamma I) \text{div} \nu = -\text{div}_{\Sigma} D\gamma + 2H\gamma, \quad (3)$$

where $H$ is the mean curvature of $X$. Hence, $\gamma$-minimal surfaces are immersed surfaces whose anisotropic mean curvature $\Lambda$ vanishes at every point. Since the first variation of the “enclosed volume” $V[X] := (1/3) \int_X \langle X, \nu \rangle \, d\Sigma$ satisfies

$$\delta V[X] = \int_{\Sigma} \langle \delta X, \nu \rangle \, d\Sigma,$$

the equation $\Lambda \equiv 0$ constant characterizes critical points of $\mathcal{F}$ with the enclosed volume constrained to be a constant. If $\Lambda$ is constant, $X$ is called a surface of constant anisotropic mean curvature. In the case where $\gamma \equiv 1$, $\Lambda = 2H$ holds.
Now we extend the function $\gamma$ in a homogeneous way to a function $\tilde{\gamma}$ as follows.

(i) $\tilde{\gamma}(X) = 0$ if and only if $X = \mathbf{0}$.

(ii) positive homogeneity of degree one:

$$\tilde{\gamma}(rX) = r\gamma(X), \quad \forall r \geq 0, \quad X \in \Omega.$$ 

In the special case where $\gamma(X) \equiv 1$, $\tilde{\gamma}(X) \equiv |X|$.

Let us consider a surface which is a graph of a $C^\infty$ function $\varphi: \Sigma \subset (\mathbb{R}^2) \rightarrow \mathbb{R}$ as follows:

$$X: \Sigma \rightarrow \mathbb{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to $X$ is given by

$$\nu = \frac{(-\varphi_1, -\varphi_2, 1)}{(1 + |D\varphi|^2)^{1/2}}, \quad (4)$$

where

$$\varphi_1 := \partial_1 \varphi, \quad \varphi_2 := \partial_2 \varphi, \quad D\varphi := (\varphi_1, \varphi_2).$$

**Lemma 1.** Set $\varphi_{ij} := \varphi_{x_i x_j}$, for $i, j = 1, 2$. Then

$$\Lambda = \sum_{i,j=1,2} \tilde{\gamma}_{x_i x_j} |_{X=(-D\varphi,1)} \varphi_{ij} \tag{5}$$

holds. In the special case where $\tilde{\gamma}(X) \equiv |X|$, the right hand side of (5) is

$$\frac{(1 + \varphi_1^2)\varphi_{11} - 2\varphi_1 \varphi_2 \varphi_{12} + (1 + \varphi_2^2)\varphi_{22}}{(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}}, \quad (6)$$

which is the twice of the mean curvature $H$ of $X$.

**Proof.** In the integrals below, we will write $\varphi(x_1, x_2)$, $((u_1, u_2) \in \Sigma)$, in order to avoid confusion. We have

$$\delta \mathcal{F} = -\int_\Sigma \tilde{\gamma}(\varphi) (\nu \cdot \varphi_{x_1}) du_1 du_2 = -\int_\Sigma \tilde{\gamma}(\varphi_{x_2}) (\nu \cdot \varphi_{x_2}) du_1 du_2,$$

$$= \int_\Sigma \frac{\partial \tilde{\gamma}}{\partial u_1} (\nu \cdot \varphi) du_1 du_2 - \int_\Sigma (\tilde{\gamma}_{x_1} \nu_1 u_1 + (\tilde{\gamma}_{x_2} \varphi_{x_2}) u_2) du_1 du_2.$$

By partial differentiation, the last term of the above equation becomes

$$\int_\Sigma (-\varphi_{x_2} \varphi_{x_2} du_1 + \varphi_{x_1} \varphi_{x_1} du_2) = 0,$$

because $\varphi_2 = 0$ on $\partial \Sigma$. Therefore,

$$\delta \mathcal{F} = -\int_\Sigma \tilde{\gamma}(\varphi_{x_1} \varphi_{x_1} + 2\varphi_{x_1} \varphi_{x_2} + \varphi_{x_2} \varphi_{x_2}) \varphi_{x_2} du_1 du_2$$

$$= -\int_\Sigma \tilde{\gamma}(\varphi_{x_2} \varphi_{x_2}) (\delta X, \nu) d\Sigma, \quad (7)$$

here we used (4) and the followings:

$$\delta X = (0, 0, \varphi_1), \quad d\Sigma = (1 + |D\varphi|^2)^{1/2} du_1 du_2.$$ 

In view of (2), (7) implies (5). By a direct computation, we obtain (6).

We will give another representation of the anisotropic mean curvature. Let $X: \Omega \rightarrow \mathbb{R}^3$ be an immersion with Gauss map $\nu$. Let $\{e_1, e_2\}$ be a locally defined frame on $S^2$ such that $(D^2 \nu)^{1/2}$ is an immersion with Gauss map $\nu$. Note that the basis $\{e_1, e_2\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent plane of $X$ at $p$. Let $(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2 \nu)^{1/2} d\nu = (-w_{11}/\mu_1 - w_{12}/\mu_1, -w_{21}/\mu_2, -w_{22}/\mu_2).$$

This with (3) gives

$$\Lambda = w_{11}/\mu_1 + w_{22}/\mu_2. \quad (8)$$

Note that $D^2 \nu + \gamma I$ is the inverse of the differential of the Gauss map of $M_\gamma$ and so its eigenvalues $1/\mu_2$ are the negatives of the reciprocals of the principal curvatures of the standard body $M_\gamma$ with respect to the outward unit normal.

For an axisymmetric $\gamma$, $\mu_1, \mu_2$ are represented in terms of $\gamma$ as follows:

**Lemma 2.** Let $\gamma: \Omega \rightarrow \mathbb{R}_+$ be a positive $C^\infty$ function on a nonempty open set $\Omega$ of the unit sphere $S^2$. Assume that the matrix $D^2 \gamma + \gamma I$ is non-singular at each point $\nu$ in $\Omega$. Assume also that $\gamma$ is axisymmetric, say $\gamma(\nu) = \gamma(\nu_1)$. Then the standard body $M_\gamma$ for $\gamma$ is also symmetric with respect to the $x_3$-axis. Denote by $\mu_1, \mu_2$ the principal curvatures of $M_\gamma$ with respect to the normal $-\nu$. We let $\mu_1$ be the curvature of the generating curve of $M_\gamma$. Then

$$\mu_1^{-1} = (1 - \nu_3^2)\gamma' + \mu_2^{-1}, \quad \mu_2^{-1} = \gamma - \nu_3 \gamma'$$

holds.

**Proof.** The proof is the same as the proof of the same formulas for the case where $\gamma$ satisfies the convexity condition which was given in [8, Section 5].

3. **Proof of Theorem 1**

In this section, we give a proof of Theorem 1 which was given in the introduction.

Denote by $\langle \cdot, \cdot \rangle_L$ the scalar product for the Minkowski metric $dx_1^2 + dx_2^2 - dx_3^2$ in $\mathbb{R}_3$. Let $X: \Sigma (\subset \mathbb{R}^2) \rightarrow \mathbb{R}_3$ be a spacelike or timelike immersed surface. Let $(u_1, u_2)$ be
local coordinates of $\Sigma$. Denote by $H_L$ the mean curvature of $X$. That is, $H_L$ is defined by

$$H_L = \frac{\bar{h}_{11} \bar{g}_{22} - 2 \bar{h}_{12} \bar{g}_{12} + \bar{h}_{22} \bar{g}_{11}}{2(\bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2)},$$

where $\bar{g}_{ij} := \langle X_{u_i}, X_{u_j} \rangle_L$, $\bar{h}_{ij} := \langle X_{u_i u_j}, \nu^L \rangle_L$ for $i, j = 1, 2$, and $\nu^L$ is the unit normal vector field along $X$ for the Minkowski metric. Let $A_L[X]$ be the area of $X$ defined by

$$A_L[X] := \int_\Sigma \pmatrix{\det(\bar{g}_{ij})} \, du_1 du_2.$$

Let $X_e$ be an arbitrary compactly-supported variation of $X$. We will compute the first variation of $A_L$ for the unit normal vector field along $X$ for the Minkowski metric.

Proof of Theorem 1. First we assume that the surface is a graph of a $C^\infty$ function $\varphi: \Sigma (\subset \mathbb{R}^2) \to \mathbb{R}$ as follows:

$$X: \Sigma \to \mathbb{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The area element $d\Sigma_L$ of $X$ is given by

$$d\Sigma_L = |1 - \varphi_1^2 - \varphi_2^2|^{1/2} \, dx_1 dx_2.$$ 

On the other hand, the unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to $X$ and the area element $d\Sigma$ of $X$ for the euclidean metric are

$$\nu = \left( -\varphi_1, -\varphi_2, 1 \right) \left( 1 + \varphi_1^2 + \varphi_2^2 \right)^{1/2}, \quad d\Sigma = (1 + \varphi_1^2 + \varphi_2^2)^{1/2} \, du_1 du_2.$$

Hence,

$$d\Sigma_L = \left( \left| 1 - \varphi_1^2 - \varphi_2^2 \right| \right)^{1/2} d\Sigma = |\nu_3^2 - \nu_1^2 - \nu_2^2|^{1/2} \, d\Sigma.$$

Therefore, by (2) and Proposition 2, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$.

Next, we consider the case where the considered surface $\Sigma$ cannot be represented as a graph like (10). It is sufficient to consider the case where the image of the Gauss map of $\Sigma$ is contained in the equator $\{(x_1, x_2, 0) \in S^2\}$. In this case, $\Sigma$ is timelike. It is proved that $\Sigma$ is represented as

$$X(s, t) = (x_1(s), x_2(s), t),$$

where $C(s) := (x_1(s), x_2(s))$ is a smooth plane curve. Denote by $\kappa$ the curvature of $C$. Note that $\gamma$ can be represented as $\gamma(\nu) = \gamma |\nu_3|$. Then from (8), $\Lambda(s, t) = \gamma(0) \kappa |\nu_3|$ holds. On the other hand, $H_L = \kappa |\nu_3|$ holds. Hence, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$.

4. PROOF OF THEOREM 2

Let $(x_1, x_2, x_3)$ be the standard coordinates in $\mathbb{R}^3$. We assume that $\gamma$ is symmetric with respect to the $x_3$-axis without loss of generality. So we can write $\gamma = \gamma(\nu_3)$. Assume that $\gamma$ is not a constant function.

Denote by $\Sigma$ the considered surface. First assume that $\Sigma$ is represented as $x_3 = \varphi(x_1, x_2)$. As in §2, we will write

$$\varphi_i := \varphi_{x_i}, \quad \varphi_{ij} := \varphi_{x_i x_j}, \quad (i, j = 1, 2).$$

By the formula (5) and a simple but long computation, we have

$$\Lambda = 2H \left( \gamma - \frac{\gamma'}{(1 + \varphi_1^2 + \varphi_2^2)^{1/2}} - \frac{\gamma''}{1 + \varphi_1^2 + \varphi_2^2} \right) + \left( \frac{\nu_{11} + \nu_{22}}{(1 + \varphi_1^2 + \varphi_2^2)^{3/2}} \right).$$

Hence, if $\Lambda = H = 0$ holds, then $\gamma''(\varphi_{11} + \varphi_{22}) = 0$ holds. Since

$$0 = H = \frac{(1 + \varphi_1^2)(\varphi_{11} - 2 \varphi_1 \varphi_{12} + (1 + \varphi_1^2) \varphi_{22})}{2(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}},$$
we obtain
\[ \gamma''(\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}) = 0. \] (11)
Consider any contour line \( \varphi(x_1(s), x_2(s)) \equiv \text{constant} \), \( s \) is arc length of the curve \( C : (x_1(s), x_2(s)) \), of \( \Sigma \). Denote by \( \kappa \) the curvature of \( C \). Then,

**Lemma 3.**

\[ |\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}| = |\kappa(\varphi_1^2 + \varphi_2^2)^{3/2} \] (12)
holds.

**Proof.** Denote by \( \varphi' \) the derivative with respect to \( s \). We differentiate \( \varphi(x_1(s), x_2(s)) \equiv \text{constant} \) with respect to \( s \) to obtain
\[ \varphi_1x'_1 + \varphi_2x'_2 = 0. \] (13)
Differentiate (13) again and use \( (x'_1, x'_2) = \kappa(-x'_2, x'_1) \) to obtain
\[ \varphi_{11}(x'_1)^2 + 2\varphi_{12}x'_1x'_2 + \varphi_{22}(x'_2)^2 = \kappa(x'_1x'_2 - \varphi_2x'_1). \] (14)
By using (13), (14), and the fact that \( (x'_1)^2 + (x'_2)^2 = 1 \), we obtain (12).

Now assume that the surface is not (a part of) a plane. We remark that it is sufficient to prove that the surface is a part of a right helicoid almost everywhere. So we assume that \( \nu \neq (0, 0, \pm 1) \) at any point in \( \Sigma \), that is, \( \varphi_1, \varphi_2 \) never coincides with \( (0, 0) \). Then, (11) combined with (12) shows that \( \gamma'' \equiv 0 \) or \( \kappa \equiv 0 \) holds. If \( \gamma'' \equiv 0 \), then, by Lemma 2, \( \mu_1 \equiv \mu_2 \) holds. This means that the standard body \( M_\gamma \) for \( \gamma \) is (a part of) a sphere, and hence \( \gamma \) is a constant function, which contradicts the assumption. Hence \( \kappa \equiv 0 \) holds, and the curve \( C \) is a straight line. Therefore, \( \Sigma \) is a ruled surface. Because only planes and right helicoids are ruled surfaces which are minimal, \( \Sigma \) is a right helicoid.

If \( \Sigma \) is represented as \( x_3 = \varphi(x_1, x_2) \) in a connected neighborhood \( U \) of a point \( P_0 \in \Sigma \), then, by the above argument, \( U \) is a part of a right helicoid \( M \). Since \( \Sigma = \Sigma \cap M \) is an open and closed subset of a connected set \( \Sigma \), \( \Sigma = \Sigma \) must hold. This means that \( \Sigma \) itself is a part of a right helicoid.

If \( \Sigma \) is not represented as a graph \( x_3 = \varphi(x_1, x_2) \) at any point, then \( \nu(P) \) is in the equator of \( S^2 \) for any \( P \in \Sigma \). Hence the Gauss curvature \( K \) of \( \Sigma \) vanishes at any point. Since \( K \equiv 0 \equiv H, \Sigma \) is a plane which is parallel to the \( x_3 \)-axis.

5. **Examples**

Let \( \gamma : \Omega \to \mathbb{R}_+ \) be an axisymmetric positive \( C^\infty \) function (say, \( \gamma(\nu) = \gamma(\nu_3) \)) on a nonempty open set \( \Omega \) in \( S^2 \). Assume that the matrix \( D^2\gamma + \gamma 1 \) is non-singular at each point \( \nu \in \Omega \).

In this section, we study a special type of cyclic surfaces, that is, surfaces foliated by circles in parallel planes which are orthogonal to the \( x_3 \)-axis. So our surfaces are represented as follows:

\[ X(\theta, t) = (r(t) \cos \theta + f(t), r(t) \sin \theta + g(t), t). \] (15)

As in Lemma 2, we denote by \( \mu_1, \mu_2 \) the principal curvatures of the standard body \( M_\gamma \) with respect to the normal \( -\nu \), here \( \mu_1 \) is the curvature of the generating curve of \( M_\gamma \).

**Lemma 4.** The anisotropic mean curvature of \( X \) in (15) is given by

\[ \Lambda = \frac{r(\nu'' + f'' \cos \theta + g'' \sin \theta) - (f' \sin \theta - g' \cos \theta)}{\mu_1 r(f' \cos \theta + g' \sin \theta)^2 + 1} \]

\[ - \frac{1}{\mu_2 r \sqrt{(f' \cos \theta + g' \sin \theta)^2 + 1}}. \] (16)

**Proof.** Let \( \nu \) be the Gauss map of \( X \) as usual. Let \( \{e_1, e_2\} \) be a locally defined frame on \( S^2 \) such that \( (D^2\gamma + \gamma 1)e_i = (1/\mu_i)e_i \). Note that the basis \( \{e_1, e_2\} \) at \( \nu(p) \) also serves as an orthogonal basis for the tangent plane of \( X \) at \( p \). As in \( \S 2 \), let \( (-w) \) be the matrix representing \( d\nu \) with respect to this basis. Then

\[ (D^2\gamma + \gamma 1)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & -w_{12}/\mu_1 \\ -w_{21}/\mu_2 & -w_{22}/\mu_2 \end{pmatrix}, \]

and

\[ \Lambda = w_{11}/\mu_1 + w_{22}/\mu_2 \] (17)
holds. So, we will compute the matrix \( (w_{ij}) \).

Let \( \nu^M = (\nu_1^M, \nu_2^M, \nu_3^M) \) be the outward pointing unit normal to \( M \). Since \( M \) is a surface of revolution, \( D^2\gamma + \gamma 1 \) has eigendirections corresponding to

\[ E_1 = (0, 0, 1) - \nu_3^M \nu^M, \quad E_2 = \nu^M \times E_1 \] (18)
as long as the normal is not vertical. \( E_1, E_2 \) define an orthonormal basis \( \{e_1, e_2\} \) on \( TS^2 \) as long as \( X \) does not intersect with the vertical axis.

Set \( g_{11} = (X_\theta, X_\theta), g_{12} = (X_\theta, X_t), g_{22} = (X_t, X_t), h_{11} = (X_\theta, \nu), h_{12} = h_{21} = (X_\theta, \nu), h_{22} = (X_t, \nu) \). And set

\[ \Delta := r' + f' \cos \theta + g' \sin \theta. \]

Then,

\[ g_{11} = r^2, \]
\[ g_{12} = -rf \sin \theta + rg' \cos \theta, \]
\[ g_{22} = (r')^2 + 2f' \cos \theta + 2f'g' \sin \theta + (f')^2 + (g')^2 + 1, \]
\[ h_{11} = \frac{-r}{\sqrt{\Delta^2 + 1}}, \]
\[ h_{12} = 0, \]
\[ h_{22} = \frac{r'' + f'' \cos \theta + g'' \sin \theta}{\sqrt{\Delta^2 + 1}}, \]
and

\[ \nu = (\nu_1, \nu_2, \nu_3) : = \frac{X_\theta \times X_t}{|X_\theta \times X_t|} = \frac{1}{\sqrt{\Delta^2 + 1}} (\cos \theta, \sin \theta, -\Delta). \]

We have

\[ \hat{E}_1 := (0, 0, 1) - \nu \nu = \frac{1}{\Delta^2 + 1} (\Delta \cos \theta, \Delta \sin \theta, 1). \]
Hence,

\[ e_1 = \frac{\hat{E}_1}{|\hat{E}_1|} = \frac{1}{\sqrt{\Delta^2 + 1}}(\Delta \cos \theta, \Delta \sin \theta, 1), \quad (I) \]

\[ e_2 = \nu \times e_1 = (\sin \theta, -\cos \theta, 0). \quad (20) \]

Now we take a coordinate transformation \( \theta(u, v), t(u, v) \) so that, at an arbitrary fixed point \((u_0, v_0)\),

\[ \frac{\partial X}{\partial u} = e_1, \quad \frac{\partial X}{\partial v} = e_2 \]

are satisfied. Then, we have

\[ \frac{\partial X}{\partial \theta} \theta_u + \frac{\partial X}{\partial t} t_u = e_1, \quad \frac{\partial X}{\partial \theta} \theta_v + \frac{\partial X}{\partial t} t_v = e_2. \quad (21) \]

Inserting \( X_\theta = (-r \sin \theta, r \cos \theta, 0) \), \( X_t = (r' \cos \theta + f', r' \sin \theta + g', 1) \), (19), and (20) to (21), we obtain

\[ J := \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} \begin{pmatrix} f' \sin \theta - g' \cos \theta \\ \sqrt{\Delta^2 + 1} \end{pmatrix}, \quad \det J > 0. \]

Let \((w_{ij}), (\tilde{w}_{ij})\) be the Weingarten mappings for \(X(u, v), X(\theta, t)\), respectively. Then,

\[ (w_{ij}) = (g_{ij})^{-1}(h_{ij}), \]

\[ (w_{ij}) = \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix}^{-1} \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} = J^{-1}(\tilde{w}_{ij})J. \]

Hence, by a computation, we obtain

\[ w_{11} = \frac{r(r' + f'' \cos \theta + g' \sin \theta) - (f' \sin \theta - g' \cos \theta)^2}{r(\Delta^2 + 1)^{3/2}}, \]

\[ w_{22} = -\frac{1}{r(\Delta^2 + 1)}. \]

This with (17) gives (16).

Now we assume that the standard body \(M_\gamma\) for \(\gamma\) is a quadric surface of revolution. Then, by homothety and translation, \(M_\gamma\) is one of the followings:

(I) a spheroid: \(x_1^2 + x_2^2 + \frac{x_3^2}{a^2} = 1\),

(II) a hyperboloid of two sheets: \(x_1^2 + x_2^2 - \frac{x_3^2}{a^2} = -1\),

(III) a hyperboloid of one sheet: \(x_1^2 + x_2^2 - \frac{x_3^2}{a^2} = 1\),

(IV) a circular paraboloid: \(x_3 = a(x_1^2 + x_2^2)\),

where \(a\) is a positive constant.

**Lemma 5.** The support functions \(\gamma\) of \(M_\gamma\) in the above (I)–(IV) are respectively given by the followings:

(I) \(\gamma(v_3) = \sqrt{1 + bv_3^2}, \quad (b := a^2 - 1 > -1)\),

(II) \(\gamma(v_3) = \sqrt{1 +bv_3^2}, \quad (b := a^2 + 1 > 1, \frac{1}{\sqrt{b}} < |v_3| \leq 1)\),

(III) \(\gamma(v_3) = \sqrt{1 - bv_3^2}, \quad (b := a^2 + 1 > 1, |v_3| < \frac{1}{\sqrt{b}})\),

(IV) \(\gamma(v_3) = \frac{-1 + v_3^2}{bv_3}, \quad (b := 4a > 0, v_3 \neq 0)\).

**Proof.** (I) Represent the upper half of \(M_\gamma\) as

\[ Y(x_1, x_2) = (x_1, x_2, a\sqrt{1 - x_1^2 - x_2^2}). \]

The outward pointing unit normal \(\nu\) to \(Y\) is given by

\[ \nu = \frac{1}{\sqrt{1 + (a^2 - 1)(x_1^2 + x_2^2)}}(ax_1, ax_2, \sqrt{1 - x_1^2 - x_2^2}). \]

Hence, we obtain

\[ \gamma = \langle Y, \nu \rangle = \sqrt{1 + (a^2 - 1)v_3^2} = \sqrt{1 + bv_3^2}, \]

which proves (I).

Similarly, we obtain (II)–(IV). \(\square\)

**Proposition 3.** Let \(\gamma\) be a function given by the above (I)–(IV). Then, there exist \(\gamma\)-minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the \(x_3\)-axis. Up to translations in \(\mathbb{R}^3\), rotations around the \(x_3\)-axis, and symmetry with respect to a plane \(\{x_3 = \text{constant}\}\), they are respectively represented as follows.

(I) Catenoid-type:

\[ X(\theta, t) = \left( \frac{\cosh(ct)}{c\sqrt{1 + b}}, \frac{\cosh(ct)}{c\sqrt{1 + b}} \sin \theta, t \right), \quad c \neq 0. \quad (22) \]

Riemann-type:

\[ X(\theta, r) = \left( r \cos \theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r \sin \theta, \right) \]

\[ \sqrt{1 + b} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, \quad (23) \]

\[ c_1 \neq 0, \quad r \geq \left( \frac{-c_2 + \sqrt{c_2^2 + 4c_1^2}}{2c_1^2} \right)^{1/2}. \quad (24) \]

(II) Catenoid-type:

\[ X(\theta, t) = \left( \frac{\sinh(ct)}{c\sqrt{b - 1}} \cos \theta, \frac{\sinh(ct)}{c\sqrt{b - 1}} \sin \theta, t \right), \quad (25) \]

\[ c \neq 0, \quad t \neq 0. \quad (26) \]

Riemann-type:

\[ X(\theta, r) = \left( r \cos \theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r \sin \theta, \right) \]

\[ \sqrt{b - 1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, \quad (27) \]

\[ c_1 \neq 0, \quad c_2 > 2|c_1|, \quad r > 0. \quad (28) \]

(III) Catenoid-type:

\[ X(\theta, t) = \left( \frac{\sin(ct)}{c\sqrt{b - 1}} \cos \theta, \frac{\sin(ct)}{c\sqrt{b - 1}} \sin \theta, t \right), \quad (29) \]

\[ c \neq 0, \quad \sin(ct) \neq 0. \quad (30) \]
\(X(\theta, r) = \left( r \cos \theta + \int \frac{c_1 r dr}{\sqrt{c_1^2 r^4 + c_2}}, r \sin \theta, \frac{\sqrt{b - 1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2}}}{c_1 \neq 0, c_2 > 0, r > 0. \right)
\)

(IV) Catenoid-type:
\[
X(\theta, t) = (e^{ct} \cos \theta, e^{ct} \sin \theta, t), \quad c \neq 0.
\]

(IV) Riemann-type:
\[
X(\theta, r) = \left( r \cos \theta + \int \frac{c_1 r dr}{\sqrt{c_1^2 r^4 + c_2}}, r \sin \theta, \frac{\sqrt{b - 1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2}}}{c_1 \neq 0, c_2 > 0, r > 0. \right)
\]

Remark 1. Let \(\gamma\) be a function given in (I), (II), or (III) in Lemma 5. Set \(\alpha := \sqrt{b - 1}\) for (I), and \(\alpha \equiv \sqrt{b - 1}\) for (II) and (III). If we take the transformation \(\hat{x}_1 = x_1, \hat{x}_2 = x_2, \hat{x}_3 = x_3/\alpha\), then an immersion \(X = (x_1, x_2, x_3)\) in \(\mathbb{R}^3\) is \(\gamma\)-minimal if and only if \(\hat{X} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)\) is a minimal surface (for (I)), a spacelike zero mean curvature surface in \(\mathbb{R}^3\) (for (II)), a timelike zero mean curvature surface in \(\mathbb{R}^3\) (for (III)), respectively. This is proved by the same way as Example 4.4 in [8].

Remark 2. \(\gamma\)-minimal surfaces for \(\gamma\) given in (IV) in Lemma 5 are graphs of harmonic functions (cf. [12]).

Proof of Proposition 3. Let \(\mu_1, \mu_2\) be the principal curvatures of \(M_\gamma\) with respect to the normal \(-\nu\) in the same way as Lemma 2.

We represent the surface as
\[
X(\theta, t) = (r(t) \cos \theta + f(t), r(t) \sin \theta + g(t), t).
\]

Note that \(X\) is a surface of revolution around the \(x_3\)-axis if and only if \(f \equiv g \equiv 0\) holds.

As in the proof of Lemma 4, we set
\[
\Delta := r' + f' \cos \theta + g' \sin \theta.
\]

Then, the Gauss map \(\nu\) of \(X\) is
\[
\nu(\theta, t) := \frac{X_\theta \times X_t}{|X_\theta \times X_t|} = \frac{1}{(\Delta^2 + 1)^{1/2}} (\cos \theta, \sin \theta, -\Delta).
\]

(I) By a simple computation using Lemma 2, we obtain
\[
\frac{1}{\mu_1} = \frac{b + 1}{(1 + b \nu_3^2)^{3/2}}, \quad \frac{1}{\mu_2} = \frac{1}{\sqrt{1 + b \nu_3^2}}.
\]

Since \(\nu_3 = \frac{-\Delta}{(\Delta^2 + 1)^{1/2}}\), we obtain
\[
\frac{1}{\mu_1} = (b + 1) \left( \frac{1 + \Delta^2}{1 + (b + 1) \Delta^2} \right)^{3/2}, \quad \frac{1}{\mu_2} = \left( \frac{1 + \Delta^2}{1 + (b + 1) \Delta^2} \right)^{1/2}.
\]

By Lemma 4 with (37) and (38), we see that \(\Lambda = 0\) if and only if
\[
(b + 1)(rf'' - 2rf'f') \cos \theta + (b + 1)(rg'' - 2rg'g') \sin \theta + (b + 1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0
\]
holds. This gives the following system of ordinary differential equations:
\[
rf'' - 2rf'f' = 0, \quad rg'' - 2rg'g' = 0, \quad (b + 1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0.
\]
From (39) and (40), we have
\[ f' = c_1 r^2, \quad g' = c_2 r^2. \] 
(42)

First, assume \( f' = g' = 0 \). Then, (41) is equivalent to
\[ (r''r - (r')^2) - \frac{1}{b+1} = 0. \] 
(43)

By a standard way, we see that the general solution of (43) is
\[ r = \frac{\cosh(c_3(t + c_4))}{c_3 \sqrt{b+1}}, \quad c_3 \neq 0, \] 
(44)
which gives the formula (22).
Next, we assume that \( f' \neq 0 \) or \( g' \neq 0 \) holds. Because of (42), \( c_2 f' - c_1 g' = 0 \) holds. This implies that \( c_2 f - c_1 g \) is constant. Since \( (c_1, c_2) \neq (0, 0) \), by rotating the surface around the \( x_3 \)-axis if necessary, we may assume that
\[ f' = c_1 r^2 (c_1 \neq 0), \quad g(t) \equiv 0 \] 
holds. Then, (41) is equivalent to
\[ 1 + (b+1)((r')^2 - r''r + c_1^2 r^4) = 0. \]
From this, by a standard argument, we obtain
\[ \frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_3 r^2 - \frac{1}{b+1}}. \] 
(46)

Hence,
\[ t = \pm \int \frac{dr}{\sqrt{c_1^2 r^4 + c_3 r^2 - \frac{1}{b+1}}} = \pm \sqrt{b+1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_3 r^2 - 1}}, \]
\[ c_6 := \sqrt{b+1} c_1 \neq 0, \quad c_7 := (b+1)c_3. \]
(47)

By using (45) and (46), we easily obtain
\[ f = \pm \int \frac{c_0 r^2 dr}{\sqrt{c_1^2 r^4 + c_3 r^2 - 1}}, \quad c_6 \neq 0. \] 
(48)

(47) with (48) gives the formula (23). Moreover,
\[ c_1^2 r^4 + c_2 r^2 - 1 \geq 0 \]
if and only if
\[ r \geq \left( -c_2 + \sqrt{c_2^2 + 4c_1^2 \over 2c_1^2} \right)^{1/2} \]
holds, which gives the condition (24).

(II) The proof is similar to the proof of (I). We have
\[ \frac{1}{\mu_1} = \frac{-b-1}{(-1 + b\nu_2^2)^{3/2}} = -(b-1) \left( \frac{1 + \Delta^2}{-1 + (b-1)\Delta^2} \right)^{3/2}, \]
holds. Then, (51) is equivalent to
\[ 1 - (b-1)((r')^2 - r''r + c_1^2 r^4) = 0. \] 
(60)

From this, by a standard argument, we obtain
\[ \frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_3 r^2 + \frac{1}{b-1}}. \] 
(61)

Hence,
\[ t = \pm \sqrt{b-1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_3 r^2 + 1}}, \]
\[ c_3 := \sqrt{b-1} c_1 \neq 0, \quad c_4 := (b-1)c_2. \]
By using \( f' = c_1 r^2 \) and (61), we obtain
\[
f = \pm \int \frac{c_3 r^2 \, dr}{\sqrt{c_1 r^4 + c_2 r^2 + 1}}, \quad c_3 \neq 0. \tag{63}
\]
(62) with (63) gives the formula (27).

Next, we will check whether the surface is compatible with \( \gamma \) or not. Note that the surface \( X \) is compatible with \( \gamma \) if and only if its Gauss map \( \nu = (\nu_1, \nu_2, \nu_3) \) satisfies \( b\nu_3^2 - 1 > 0 \) for all \( \theta \).

For the surface \( X \) given by (56), the Gauss map \( \nu \) is
\[
\nu := \frac{X_\theta \times X_r}{|X_\theta \times X_r|} = b^{-1/2} \left( \sqrt{b - 1} \cos \theta, \sqrt{b - 1} \sin \theta, -1 \right).
\]
This shows that \( b\nu_3^2 - 1 \equiv 0 \), and hence \( X \) is not compatible with \( \gamma \).

For the surface \( X \) given by (57),
\[
\nu = \left(1 + \frac{\cosh^2(ct)}{b - 1}\right)^{-1/2} \left( \cos \theta, \sin \theta, -\frac{\cosh(ct)}{\sqrt{b - 1}} \right).
\]
This shows that \( b\nu_3^2 - 1 \geq 0 \) always holds, and that \( b\nu_3^2 - 1 > 0 \) for all \( \theta \) if and only if \( t \neq 0 \).

For the surface \( X \) given by (58),
\[
\nu = \left(1 + \frac{\cos^2(ct)}{b - 1}\right)^{-1/2} \left( \cos \theta, \sin \theta, -\frac{\cos(ct)}{\sqrt{b - 1}} \right). \tag{64}
\]
This shows that \( b\nu_3^2 - 1 \leq 0 \) always holds, and hence \( X \) is not compatible with \( \gamma \).

For the surface \( X \) given by (27), the Gauss map \( \nu \) is
\[
\nu := \frac{X_\theta \times X_r}{|X_\theta \times X_r|} \tag{65},
\]
\[
X_\theta \times X_r = \left( \sqrt{b - 1} \cos \theta, \sqrt{b - 1} \sin \theta, \frac{c_1 r^2 \cos \theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}} \right) \tag{66}.
\]
This shows that \( b\nu_3^2 - 1 > 0 \) for all \( \theta \) if and only if \( c_2 > 2|c_1| \) holds.

(III) The proof is again similar to the proof of (I). We see that the condition \( \Delta = 0 \) is equivalent to the condition that the system of ordinary differential equations (49), (50) and (51) holds.

First, assume \( f' = g' = 0 \). Then, (51) is equivalent to (52). Note that \( b - 1 > 0 \). The general solutions of (52) are given by (53), (54), and (55), and corresponding surfaces are given by (56), (57), and (58). Note that the surface \( X \) is compatible with \( \gamma \) if and only if its Gauss map \( \nu = (\nu_1, \nu_2, \nu_3) \) satisfies \( 1 - b\nu_3^2 > 0 \) for all \( \theta \). As we have seen above, for the surfaces (56) and (57), \( b\nu_3^2 - 1 \geq 0 \) holds at every point. Hence, they are not compatible with \( \gamma \). Therefore, only the possibility is the case (58), which is the same as the formula (29). The Gauss map \( \nu \) for this surface is given by (64), which shows that, \( 1 - b\nu_3^2 > 0 \) for all \( \theta \) if and only if \( \sin(ct) \neq 0 \).

Next, we assume that \( f' \neq 0 \) or \( g' \neq 0 \) holds. By rotating the surface around the \( x_3 \)-axis if necessary, we may assume that \( f \) and \( g \) satisfy (59). Then, (51) is equivalent to (60). Hence, we obtain the formula (31).

We will check whether the surface is compatible with \( \gamma \) or not. The Gauss map \( \nu \) is given by (65), (66) as in the case (II). This shows that, \( 1 - b\nu_3^2 > 0 \) for all \( \theta \) if and only if both \( c_2 < -2|c_1| \) and
\[
0 < r \leq \left( \frac{|c_2|}{\sqrt{c_1^2 r^4 + c_2 r^2}} \right)^{1/2}
\]
holds, which gives the condition (32).

(IV) The proof is again similar to the proof of (I). We obtain
\[
\frac{1}{\mu_1} = -\frac{2}{b\nu_3}, \quad \frac{2(\Delta^2 + 1)^{3/2}}{b\Delta \Sigma}, \tag{67}
\]
\[
\frac{1}{\mu_2} = -\frac{2}{b\nu_3} = \frac{2(\Delta^2 + 1)^{1/2}}{b\Delta}. \tag{68}
\]
By Lemma 4 with (67) and (68), we see that \( \Lambda = 0 \) if and only if
\[
(r f'' - 2r' f') \cos \theta + (r g'' - 2r' g') \sin \theta
+ (r'' r - (r')^2 - (f')^2 - (g')^2) = 0
\]
holds. This gives the following system of ordinary differential equations:
\[
r f'' - 2r' f' = 0, \tag{69}
\]
\[
r g'' - 2r' g' = 0, \tag{70}
\]
\[
r'' r - (r')^2 - (f')^2 - (g')^2 = 0. \tag{71}
\]
From (69) and (70), we have
\[
f' = c_1 r^2, \quad g' = c_2 r^2. \tag{72}
\]
When \( f' = g' = 0 \) holds, (71) is equivalent to
\[
r'' r - (r')^2 = 0. \tag{73}
\]
The general solution of (73) is
\[
r = e^{c_1 t + c_2},
\]
which gives the formula (33).

When \( f' \neq 0 \) or \( g' \neq 0 \) holds, by rotating the surface around the \( x_3 \)-axis if necessary, we may assume that
\[
f' = c_1 r^2 \quad (c_1 \neq 0), \quad g(t) \equiv 0 \tag{74}
\]
holds. Then, (71) is equivalent to
\[
(r')^2 - r'' r + c_1^2 r^4 = 0. \tag{75}
\]
From this, by a standard argument, we obtain
\[
\frac{dr}{dt} = \pm \sqrt{c_1^2 r^4 + c_2 r^2}. \tag{76}
\]
Hence,
\[ t = \pm \int \frac{dr}{\sqrt{c_1 r^2 + c_2}} \]  
(77)

By using \( f' = c_1 r^2 \) and (76), we obtain
\[ f = \pm \int \frac{c_1 r}{\sqrt{c_1 r^2 + c_2}} dr, \quad c_1 \neq 0. \]  
(78)

(77) with (78) gives the formula (34).

Next, we will check whether the surface is compatible with \( \gamma \) or not. Note that the surface \( X \) is compatible with \( \gamma \) if and only if its Gauss map \( \nu = (\nu_1, \nu_2, \nu_3) \) satisfies \( \nu_3 \neq 0 \) for all \( \theta \).

For the surface \( X \) given by (33), the Gauss map \( \nu \) is
\[
\nu := \frac{X_t \times X_r}{|X_t \times X_r|} = \left( 1 + c^2 e^{2ct} \right)^{-1/2} \left( \cos \theta, \sin \theta, -ce^{ct} \right).
\]
This shows that, \( \nu_3 \neq 0 \) for all \( \theta \) if and only if \( c \neq 0 \).

For the surface \( X \) given by (34), the Gauss map \( \nu \) is
\[
\nu := \frac{X_{\theta} \times X_r}{|X_{\theta} \times X_r|},
\]
\[
X_{\theta} \times X_r = \left( \cos \theta, \frac{c_1 r \cos \theta}{(c_1 r^2 + c_2)^{1/2}}, \frac{c_1 r \cos \theta}{(c_1 r^2 + c_2)^{1/2}} - r \left( 1 + \frac{c_1 r^2 + c_2}{(c_1 r^2 + c_2)^{1/2}} \right) \right).
\]
This shows that, \( \nu_3 \neq 0 \) for all \( \theta \) if and only if \( c_2 > 0 \) holds.

\[ \square \]

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