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Non-convex anisotropic surface energy and zero mean curvature surfaces in the Lorentz-Minkowski space

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Abstract. An anisotropic surface energy functional is the integral of an energy density function over a surface. The energy density depends on the surface normal at each point. The usual area functional is a special case of such a functional. We study stationary surfaces of anisotropic surface energies in the euclidean three-space which are called anisotropic minimal surfaces. For any axisymmetric anisotropic surface energy, we show that, a surface is both a minimal surface and an anisotropic minimal surface if and only if it is a right helicoid. We also construct new examples of anisotropic minimal surfaces, which include zero mean curvature surfaces in the three-dimensional Lorentz-Minkowski space as special cases.

Keywords. anisotropic, mean curvature, minimal surface, zero mean curvature surface, Lorentz-Minkowski space, Wulff shape

1. Introduction

Let $\gamma\colon \Omega \to \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the two-dimensional unit sphere $S^2 := \{X \in \mathbf{R}^3 \ ; \ |X| = 1\}$. Let $X\colon \Sigma \to \mathbf{R}^3$ be an immersion from a two-dimensional oriented connected compact C^∞ manifold Σ (with or without boundary) to the three-dimensional euclidean space $\mathbf{R}^3$. Denote by $\nu = (\nu_1, \nu_2, \nu_3)\colon \Sigma \to S^2$ the unit normal along X (in other words, the Gauss map of X). If $\nu(\Sigma) \subset \Omega$, we say that X is compatible with γ and we define the following functional.

$$\mathcal{F}[X] = \int_\Sigma \gamma(\nu)\, d\Sigma\,, \tag{1}$$

where $d\Sigma$ is the area element of X. Such a functional is used to model anisotropic surface energies. Applications can be found in many branches of the physical sciences including metallurgy and crystallogrophy ([14, 15]). We will call $\mathcal{F}[X]$ the anisotropic energy of X, and γ the energy density function.

We call stationary surfaces of (1) for compactly-supported variations γ-minimal surfaces. It is obvious that, for $\gamma \equiv 1$, γ-minimal surfaces are usual minimal surfaces.

Denote by $D\gamma$ and $D^2\gamma$ the gradient and the Hessian of γ on Ω, respectively. Denote by 1 the identity endomorphism field on the tangent space $T_\nu(S^2)$. If the matrix $D^2\gamma+\gamma 1$ is non-singular at each point ν in Ω, a mapping $Y\colon \Omega \to \mathbf{R}^3$ defined by $Y(\nu) = D\gamma + \gamma(\nu)\nu$ is an immersion and Y defines the uniquely determined immersed surface with unit normal ν whose support function coincides with γ, that is $\gamma(\nu) = \langle Y(\nu), \nu\rangle$ holds. We say that Y is the standard body for γ. (As for the terminology "standard body", we quote [12].) We will sometimes use the symbol M_γ to represent the mapping Y or the image $Y(\Omega)$ of Y.

We say that $\gamma\colon \Omega \to \mathbf{R}_+$ satisfies the *convexity condition*, if the matrix $D^2\gamma + \gamma 1$ is positive definite at each point ν in Ω. In this case, the standard body M_γ for γ is strongly convex (that is, the principal curvatures of M_γ are positive everywhere), and the functional $\mathcal{F}$ appearing in (1) is called a constant coefficient parametric elliptic functional, and stationary surfaces are extensively studied in recent years.

In this paper, we do not assume the convexity condition. By this generalization, we obtain a more variety of important examples. For example, zero mean curvature immersions in the Lorentz-Minkowski space $\mathbf{R}^3_1 := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ ; \ ds^2 = dx_1^2 + dx_2^2 - dx_3^2\}$ arise as γ-minimal surfaces for a certain simple function γ as follows (cf. §3).

Theorem 1. *Set* $\Omega_1 := \{\nu = (\nu_1, \nu_2, \nu_3) \in S^2 \ ; \ |\nu_3| > \sqrt{2}/2\}$, $\Omega_2 := \{\nu \in S^2 \ ; \ |\nu_3| < \sqrt{2}/2\}$. *Define a function* $\gamma\colon S^2 \to \mathbf{R}$ *as* $\gamma(\nu) = \sqrt{|\nu_3^2 - \nu_1^2 - \nu_2^2|} = \sqrt{|2\nu_3^2 - 1|}$. *Then, an immersion* $X\colon \Sigma \to \mathbf{R}^3$ *with Gauss image* $\nu(\Sigma) \subset \Omega_1 \cup \Omega_2$ *is* γ*-minimal if and only if the mean curvature of* X *is zero as an immersed surface in* $\mathbf{R}^3_1$.

This result indicates that the recent investigations about zero mean curvature surfaces in $\mathbf{R}^3_1$ changing their causal type across null curves (regular curves whose velocity vector fields are lightlike) or lightlike lines from spacelike zero mean curvature surfaces to timelike zero mean curvature surfaces ([3, 6, 5, 4, 2]) should be very natural and reasonable. Probably the most well-known example of such surfaces is the right helicoid with the timelike axis as its axis, which changes its causal type across a null curve from a spacelike zero mean curvature surface to a timelike zero mean curvature surface ([3, 6]). In §4, we will show a more general remarkable result as follows.

Theorem 2. *Let $\gamma\colon \Omega \to \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω in S^2. Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω. Assume also that γ is axisymmetric and not a constant function. Let $X\colon \Sigma \to \mathbf{R}^3$ be an immersion which is compatible with γ. Then, X is both minimal and γ-minimal if and only if it is a part of either a plane or a right helicoid whose axis is parallel to the axis of γ.*

This result is a generalization of [7, Theorem 4.2] and a refinement of [9, Proposition III.1]. [7, Theorem 4.2] proves that a spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the timelike axis are only both a minimal surface in the euclidean space $\mathbf{R}^3$ and a spacelike zero mean curvature surface in $\mathbf{R}^3_1$. [9, Proposition III.1] proves that a right helicoid is a γ-minimal surface for any axisymmetric γ whose axis is parallel to the axis of the helicoid itself.

Theorem 2 combined with Theorem 1 implies the following:

Corollary 1. *A spacelike plane and the spacelike part of a right helicoid whose axis is parallel to the x_3-axis are only both a minimal surface in the euclidean space $\mathbf{R}^3$ and a spacelike zero mean curvature surface in $\mathbf{R}^3_1$. Also, a timelike plane and the timelike part of a right helicoid whose axis is parallel to the x_3-axis are only both a minimal surface in $\mathbf{R}^3$ and a timelike zero mean curvature surface in $\mathbf{R}^3_1$.*

In general, it is not easy to construct examples of γ-minimal surfaces. For any axisymmetric energy density function γ, there exist γ-minimal surfaces which are also symmetric with respect to the same axis as γ. The existence theorem and a certain kind of representation formula of these surfaces were given in [8] and they were called anisotropic catenoid. Although the convexity condition for γ was assumed in [8], the method there works also for non convex γ. In this paper, for certain classes of γ, we will give another type of examples of γ-minimal surfaces which are foliated by parallel circles but are not surfaces of revolution. We will call them γ-minimal surfaces of Riemann-type after Riemann's minimal surfaces in $\mathbf{R}^3$.

Proposition 1. *Let $\gamma\colon \Omega \to \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω in S^2. Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point $\nu \in \Omega$. We also assume that the standard body M_γ for γ is a quadric surface of revolution. Then, there are γ-minimal surfaces of Riemann-type.*

From Theorem 1, we see that spacelike and timelike zero mean curvature surfaces of Riemann-type in $\mathbf{R}^3_1$ are obtained as special cases of surfaces given by Proposition 1. Actually, for $\gamma|_{\Omega_1}$ in Theorem 1, M_γ is a hyperboloid of two sheets, and for $\gamma|_{\Omega_2}$, M_γ is a hyperboloid of one sheet (§5, Lemma 5).

We should remark that zero mean curvature surfaces of Riemann-type in $\mathbf{R}^3_1$ were studied also in [10, 11].

In §5, for γ satisfying the assumption in Proposition 1, we will give explicit parameter representations of all γ-minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the rotation axis of M_γ (Proposition 3). Actually, Proposition 1 is a corollary of Proposition 3.

Some of the results in this article can be generalized to hypersurfaces in $\mathbf{R}^{n+1}$.

2. Preliminaries

In this section, we give the definitions of the Wulff shape, anisotropic mean curvature, and their fundamental properties and representation formulas. We quote [12, 1, 8] as references.

Let $\gamma\colon \Omega \to \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the unit sphere S^2. Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω.

If $\Omega = S^2$, then, for any $V > 0$, there exists a uniquely determined (up to translations in $\mathbf{R}^3$) convex surface $W(V)$ such that $W(V)$ attains the minimum of $\mathcal{F}$ among all closed piecewise smooth surfaces in $\mathbf{R}^3$ enclosing the 3-dimensional volume V ([13]). For the special value $V_0 := (1/3)\int_{S^2} \gamma(\nu)\, dS^2$, $W(V_0)$ is called the Wulff shape for γ, and we will denote it by W. In the special case where $\gamma \equiv 1$, $\mathcal{F}[X]$ is the usual area of the surface X and W is the unit sphere S^2. In general, W is not smooth. W is a smooth strongly convex surface if and only if γ satisfies the convexity condition (see §1). In this case, W can be parametrized by the smooth mapping

$$Y\colon S^2 \to \mathbf{R}^3, \qquad Y(\nu) = D\gamma + \gamma(\nu)\nu,$$

where we regard $D\gamma$ at $\nu \in S^2$ as a point in $\mathbf{R}^3$ in the canonical manner. We remark that the outward unit normal to W at point $Y(\nu)$ coincides with ν. And the function γ coincides with the support function of W, that is $\gamma(\nu) = \langle Y(\nu), \nu\rangle$, where $\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^3$. This means that W is the standard body for γ.

Let $X\colon \Sigma \to \mathbf{R}^3$ be an immersion. By parallel translation in $\mathbf{R}^3$, $D\gamma$ may be considered as a smooth tangent vector field along X. Let $X_\epsilon = X + \epsilon\delta X + \mathcal{O}(\epsilon^2)$ be a smooth, compactly supported variation of X. The anisotropic mean curvature Λ of X is defined by the first variation formula ([8])

$$\delta\mathcal{F} := \partial_\epsilon \mathcal{F}[X_\epsilon]_{\epsilon=0} = -\int_\Sigma \Lambda\langle \delta X, \nu\rangle\, d\Sigma\,, \qquad (2)$$

$$\Lambda := -\mathrm{trace}_\Sigma(D^2\gamma + \gamma 1)d\nu = -\mathrm{div}_\Sigma D\gamma + 2H\gamma, \qquad (3)$$

where H is the mean curvature of X. Hence, γ-minimal surfaces are immersed surfaces whose anisotropic mean curvature Λ vanishes at every point. Since the first variation of the "enclosed volume" $V[X] := (1/3)\int_\Sigma \langle X, \nu\rangle\, d\Sigma$ satisfies

$$\delta V[X] = \int_\Sigma \langle \delta X, \nu\rangle\, d\Sigma\,,$$

the equation $\Lambda \equiv$ constant characterizes critical points of $\mathcal{F}$ with the enclosed volume constrained to be a constant. If Λ is constant, X is called a surface of constant anisotropic mean curvature. In the case where $\gamma \equiv 1$, $\Lambda = 2H$ holds.

Now we extend the function γ in a homogeneous way to a function $\tilde{\gamma}$ as follows.

(i) $\tilde{\gamma}(X) = 0$ if and only if $X = \mathbf{0}$.

(ii) positive homogeneity of degree one:

$$\tilde{\gamma}(rX) = r\gamma(X), \quad \forall r \geq 0,\ X \in \Omega.$$

In the special case where $\gamma(X) \equiv 1$, $\tilde{\gamma}(X) \equiv |X|$.

Let us consider a surface which is a graph of a C^∞ function $\varphi\colon \Sigma\ (\subset \mathbf{R}^2) \to \mathbf{R}$ as follows:

$$X\colon \Sigma \to \mathbf{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to X is given by

$$\nu = \frac{(-\varphi_1, -\varphi_2, 1)}{(1 + |D\varphi|^2)^{1/2}}, \tag{4}$$

where

$$\varphi_1 := \varphi_{x_1}, \quad \varphi_2 := \varphi_{x_2}, \quad D\varphi := (\varphi_1, \varphi_2).$$

Lemma 1. *Set $\varphi_{ij} := \varphi_{x_i x_j}$ for $i, j = 1, 2$. Then*

$$\Lambda = \sum_{i,j=1,2} \tilde{\gamma}_{x_i x_j}\Big|_{X=(-D\varphi,1)} \varphi_{ij} \tag{5}$$

holds. In the special case where $\tilde{\gamma}(X) \equiv |X|$, the right hand side of (5) is

$$\frac{(1+\varphi_2^2)\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + (1+\varphi_1^2)\varphi_{22}}{(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}}, \tag{6}$$

which is the twice of the mean curvature H of X.

Proof. In the integrals below, we will write $\varphi(u_1, u_2)$, $((u_1, u_2) \in \Sigma)$, in order to avoid confusion. We have

$$\begin{aligned} \mathcal{F}[X] &= \iint_\Sigma \gamma(\nu)(1 + \varphi_1^2 + \varphi_2^2)^{1/2}\, du_1 du_2 \\ &= \iint_\Sigma \tilde{\gamma}\big((-\varphi_1, -\varphi_2, 1)\big)\, du_1 du_2. \end{aligned}$$

Let $X_\epsilon = (x_1, x_2, \varphi(\epsilon, x_1, x_2))$ be an arbitrary compactly-supported variation of X. We will compute the first variation of $\mathcal{F}$. We may suppose that Σ is the support of X_ϵ and $X_\epsilon|_{\partial\Sigma} = X|_{\partial\Sigma}$ holds. We compute

$$\begin{aligned} \delta\mathcal{F} &= \iint_\Sigma \Big(\tilde{\gamma}\big((-\varphi_1, -\varphi_2, 1)\big)\Big)_\epsilon\, du_1 du_2 \\ &= \iint_\Sigma \tilde{\gamma}_{x_1} \cdot (-\varphi_{1\epsilon}) + \tilde{\gamma}_{x_2} \cdot (-\varphi_{2\epsilon})\, du_1 du_2 \\ &= \iint_\Sigma \frac{\partial \tilde{\gamma}_{x_1}|_{(-D\varphi,1)}}{\partial u_1}\varphi_\epsilon + \frac{\partial \tilde{\gamma}_{x_2}|_{(-D\varphi,1)}}{\partial u_2}\varphi_\epsilon\, du_1 du_2 \\ &\quad - \iint_\Sigma (\tilde{\gamma}_{x_1}\varphi_\epsilon)_{u_1} + (\tilde{\gamma}_{x_2}\varphi_\epsilon)_{u_2}\, du_1 du_2. \end{aligned}$$

By the partial differentiation, the last term of the above equation becomes

$$\int_{\partial\Sigma} (-\tilde{\gamma}_{x_2}\varphi_\epsilon\, du_1 + \tilde{\gamma}_{x_1}\varphi_\epsilon\, du_2) = 0,$$

because $\varphi_\epsilon = 0$ on $\partial\Sigma$. Therefore,

$$\begin{aligned} \delta\mathcal{F} &= -\iint_\Sigma (\tilde{\gamma}_{x_1x_1}\varphi_{11} + 2\tilde{\gamma}_{x_1x_2}\varphi_{12} + \tilde{\gamma}_{x_2x_2}\varphi_{22})\varphi_\epsilon\, du_1 du_2 \\ &= -\iint_\Sigma \Big(\sum_{i,j=1,2} \tilde{\gamma}_{x_i x_j}\Big|_{X=(-D\varphi,1)} \varphi_{ij}\Big)\langle \delta X, \nu\rangle\, d\Sigma, \end{aligned} \tag{7}$$

here we used (4) and the followings:

$$\delta X = (0, 0, \varphi_\epsilon), \quad d\Sigma = (1 + |D\varphi|^2)^{1/2} du_1 du_2.$$

In view of (2), (7) implies (5). By a direct computation, we obtain (6). □

We will give another representation of the anisotropic mean curvature. Let $X\colon \Sigma \to \mathbf{R}^3$ be an immersion with Gauss map ν. Let $\{e_1, e_2\}$ be a locally defined frame on S^2 such that $(D^2\gamma + \gamma 1)e_i = (1/\mu_i)e_i$. Note that the basis $\{e_1, e_2\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent plane of X at p. Let $(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2\gamma + \gamma 1)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & -w_{12}/\mu_1 \\ -w_{21}/\mu_2 & -w_{22}/\mu_2 \end{pmatrix}.$$

This with (3) gives

$$\Lambda = w_{11}/\mu_1 + w_{22}/\mu_2. \tag{8}$$

Note that $D^2\gamma + \gamma 1$ is the inverse of the differential of the Gauss map of M_γ and so its eigenvalues $1/\mu_j$ are the negatives of the reciprocals of the principal curvatures of the standard body M_γ with respect to the outward unit normal.

For an axisymmetric γ, μ_i's are represented in terms of γ as follows:

Lemma 2. *Let $\gamma\colon \Omega \to \mathbf{R}_+$ be a positive C^∞ function on a nonempty open set Ω of the unit sphere S^2. Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point ν in Ω. Assume also that γ is axisymmetric, say $\gamma(\nu) = \gamma(\nu_3)$. Then the standard body M_γ for γ is also symmetric with respect to the x_3-axis. Denote by μ_1, μ_2 the principal curvatures of M_γ with respect to the normal $-\nu$. We let μ_1 be the curvature of the generating curve of M_γ. Then*

$$\mu_1^{-1} = (1 - \nu_3^2)\gamma'' + \mu_2^{-1}, \quad \mu_2^{-1} = \gamma - \nu_3\gamma' \tag{9}$$

holds.

Proof. The proof is the same as the proof of the same formulas for the case where γ satisfies the convexity condition which was given in [8, Section 5]. □

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1 which was given in the introduction.

Denote by $\langle\ ,\ \rangle_L$ the scalar product for the Minkowski metric $dx_1^2 + dx_2^2 - dx_3^2$ in $\mathbf{R}_1^3$. Let $X\colon \Sigma\ (\subset \mathbf{R}^2) \to \mathbf{R}_1^3$ be a spacelike or timelike immersed surface. Let (u_1, u_2) be local coordinates of Σ. Denote by H_L the mean curvature of X. That is, H_L is defined by

$$H_L = \frac{\tilde{h}_{11}\tilde{g}_{22} - 2\tilde{h}_{12}\tilde{g}_{12} + \tilde{h}_{22}\tilde{g}_{11}}{2(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2)},$$

where $\tilde{g}_{ij} := \langle X_{u_i}, X_{u_j}\rangle_L$, $\tilde{h}_{ij} := \langle X_{u_i u_j}, \nu^L\rangle_L$ for $i,j = 1,2$, and ν^L is the unit normal vector field along X for the Minkowski metric. Let $A_L[X]$ be the area of X defined by

$$A_L[X] := \int_\Sigma d\Sigma_L, \qquad (d\Sigma_L := |\det(\tilde{g}_{ij})|\, du_1 du_2).$$

Let X_ϵ be an arbitrary compactly-supported variation of X. We will compute the first variation of A_L. We may suppose that Σ is the support of X_ϵ and $X_\epsilon|_{\partial\Sigma} = X|_{\partial\Sigma}$ holds. Set the variation vector field as

$$\delta X := \partial_\epsilon (X_\epsilon)_{\epsilon=0} = \xi + f\nu^L, \qquad \left(\xi = \sum_{i=1,2} \xi^i X_{u_i}\right).$$

Then we have the following.

Proposition 2. *In the above setting, it holds that*

$$\partial_\epsilon A_L[X_\epsilon]_{\epsilon=0} = -2\int_\Sigma f H_L\, d\Sigma_L.$$

Proof. We here give a proof in the case where X is timelike. By a similar argument, we can prove this in the case where X is spacelike. We have

$$A_L[X_\epsilon] = \int_\Sigma \sqrt{-\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon + (\tilde{g}_{12}^\epsilon)^2}\, du_1 du_2,$$

where $\tilde{g}_{ij}^\epsilon = \langle (X_\epsilon)_{u_i}, (X_\epsilon)_{u_j}\rangle_L$ for $i,j = 1,2$. Then,

$$\begin{aligned}
\partial_\varepsilon A_L[X_\epsilon] &= \int_\Sigma \partial_\varepsilon \left(\sqrt{-\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon + (\tilde{g}_{12}^\epsilon)^2}\right) du_1 du_2 \\
&= \int_\Sigma \frac{\partial_\varepsilon\left(-\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon + (\tilde{g}_{12}^\epsilon)^2\right)}{2\sqrt{-\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon + (\tilde{g}_{12}^\epsilon)^2}}\, du_1 du_2 \\
&= \int_\Sigma \frac{\partial_\varepsilon\left(\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon - (\tilde{g}_{12}^\epsilon)^2\right)}{2\left(\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon - (\tilde{g}_{12}^\epsilon)^2\right)}\, d\Sigma_L \\
&= \int_\Sigma \frac{\tilde{g}_{22}^\epsilon\, \partial_\varepsilon \tilde{g}_{11}^\epsilon + \tilde{g}_{11}^\epsilon\, \partial_\varepsilon \tilde{g}_{22}^\epsilon - 2\tilde{g}_{12}^\epsilon\, \partial_\varepsilon \tilde{g}_{12}^\epsilon}{2\left(\tilde{g}_{11}^\epsilon \tilde{g}_{22}^\epsilon - (\tilde{g}_{12}^\epsilon)^2\right)}\, d\Sigma_L
\end{aligned}$$

holds. By a direct calculation, we have

$$\partial_\varepsilon(\tilde{g}_{ij}^\epsilon)_{\epsilon=0} = \langle \xi_{u_i}, X_{u_j}\rangle_L + \langle X_{u_i}, \xi_{u_j}\rangle_L - 2f\tilde{h}_{ij}$$

for $i,j = 1,2$. Applying the divergence theorem, it follows that

$$\begin{aligned}
\partial_\varepsilon A_L[X_\epsilon]_{\epsilon=0} &= \int_\Sigma \sum_{i=1,2} \left(\tilde{g}^{ij}\langle \xi_{u_i}, X_{u_j}\rangle_L - f\tilde{g}^{ij}\tilde{h}_{ij}\right) d\Sigma_L \\
&= \int_\Sigma (\mathrm{div}\xi - 2fH_L)\, d\Sigma_L = -2\int_\Sigma fH_L\, d\Sigma_L,
\end{aligned}$$

where we denote by $(\tilde{g}^{ij})$ the inverse matrix of $(\tilde{g}_{ij})$. □

Proof of Theorem 1. First we assume that the surface is a graph of a C^∞ function $\varphi\colon \Sigma\ (\subset \mathbf{R}^2) \to \mathbf{R}$ as follows:

$$X\colon \Sigma \to \mathbf{R}^3, \quad X(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)). \tag{10}$$

The area element $d\Sigma_L$ of X is given by

$$d\Sigma_L = |1 - \varphi_1^2 - \varphi_2^2|^{1/2}\, dx_1 dx_2.$$

On the other hand, the unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ to X and the area element $d\Sigma$ of X for the euclidean metric are

$$\nu = \frac{(-\varphi_1, -\varphi_2, 1)}{(1 + \varphi_1^2 + \varphi_2^2)^{1/2}}, \quad d\Sigma = (1 + \varphi_1^2 + \varphi_2^2)^{1/2}\, du_1 du_2.$$

Hence,

$$d\Sigma_L = \left(\frac{|1 - \varphi_1^2 - \varphi_2^2|}{1 + \varphi_1^2 + \varphi_2^2}\right)^{1/2} d\Sigma = |\nu_3^2 - \nu_1^2 - \nu_2^2|^{1/2}\, d\Sigma.$$

Therefore, by (2) and Proposition 2, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$.

Next, we consider the case where the considered surface Σ cannot be represented as a graph like (10). It is sufficient to consider the case where the image of the Gauss map of Σ is contained in the equator $\{(x_1, x_2, 0) \in S^2\}$. In this case, Σ is timelike. It is proved that Σ is represented as

$$X(s,t) = (x_1(s), x_2(s), t),$$

where $C(s) := (x_1(s), x_2(s))$ is a smooth plane curve. Denote by κ the curvature of C. Note that γ can be represented as $\gamma(\nu) = \gamma(\nu_3)$. Then from (8), $\Lambda(s,t) = \gamma(0)\kappa(s)$ holds. On the other hand, $H_L = \kappa/2$ holds. Hence, $\Lambda \equiv 0$ if and only if $H_L \equiv 0$. □

4. Proof of Theorem 2

Let (x_1, x_2, x_3) be the standard coordinates in $\mathbf{R}^3$. We assume that γ is symmetric with respect to the x_3-axis without loss of generality. So we can write $\gamma = \gamma(\nu_3)$. Assume that γ is not a constant function.

Denote by Σ the considered surface. First assume that Σ is represented as $x_3 = \varphi(x_1, x_2)$. As in §2, we will write

$$\varphi_i := \varphi_{x_i}, \quad \varphi_{ij} := \varphi_{x_i x_j}, \quad (i,j = 1,2).$$

By the formula (5) and a simple but long computation, we have

$$\begin{aligned}
\Lambda = 2H\Big(\gamma - \frac{\gamma'}{(1 + \varphi_1^2 + \varphi_2^2)^{1/2}} - \frac{\gamma''}{1 + \varphi_1^2 + \varphi_2^2}\Big) \\
+ \frac{\gamma''(\varphi_{11} + \varphi_{22})}{(1 + \varphi_1^2 + \varphi_2^2)^{3/2}}.
\end{aligned}$$

Hence, if $\Lambda = H = 0$ holds, then $\gamma''(\varphi_{11} + \varphi_{22}) = 0$ holds. Since

$$0 = H = \frac{(1 + \varphi_2^2)\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + (1 + \varphi_1^2)\varphi_{22}}{2(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}},$$

we obtain

$$\gamma''(\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}) = 0. \tag{11}$$

Consider any contour line $\varphi(x_1(s), x_2(s)) \equiv$ constant, (s is arc length of the curve $C\colon (x_1(s), x_2(s))$), of Σ. Denote by κ the curvature of C. Then,

Lemma 3.

$$|\varphi_2^2\varphi_{11} - 2\varphi_1\varphi_2\varphi_{12} + \varphi_1^2\varphi_{22}| = |\kappa|(\varphi_1^2 + \varphi_2^2)^{3/2} \tag{12}$$

holds.

Proof. Denote by " $'$ " the derivative with respect to s. We differentiate $\varphi(x_1(s), x_2(s)) \equiv$ constant with respect to s to obtain

$$\varphi_1 x_1' + \varphi_2 x_2' = 0. \tag{13}$$

Differentiate (13) again and use $(x_1'', x_2'') = \kappa(-x_2', x_1')$ to obtain

$$\varphi_{11}(x_1')^2 + 2\varphi_{12}x_1'x_2' + \varphi_{22}(x_2')^2 = \kappa(\varphi_1 x_2' - \varphi_2 x_1'). \tag{14}$$

By using (13), (14), and the fact that $(x_1')^2 + (x_2')^2 = 1$, we obtain (12). □

Now we assume that the surface is not (a part of) a plane. We remark that it is sufficient to prove that the surface is a part of a right helicoid almost everywhere. So we assume that $\nu \neq (0, 0, \pm 1)$ at any point in Σ, that is (φ_1, φ_2) never coincides with $(0, 0)$. Then, (11) combined with (12) shows that $\gamma'' \equiv 0$ or $\kappa \equiv 0$ holds. If $\gamma'' \equiv 0$, then, by Lemma 2, $\mu_1 \equiv \mu_2$ holds. This means that the standard body M_γ for γ is (a part of) a sphere, and hence γ is a constant function, which contradicts the assumption. Hence $\kappa \equiv 0$ holds, and the curve C is a straight line. Therefore, Σ is a ruled surface. Because only planes and right helicoids are ruled surfaces which are minimal, Σ is a right helicoid.

If Σ is represented as $x_3 = \varphi(x_1, x_2)$ in a connected neighborhood U of a point $P_0 \in \Sigma$, then, by the above argument, U is a part of a right helicoid M. Since $\Sigma_1 := \Sigma \cap M$ is an open and closed subset of a connected set Σ, $\Sigma_1 = \Sigma$ must hold. This means that Σ itself is a part of a right helicoid.

If Σ is not represented as a graph $x_3 = \varphi(x_1, x_2)$ at any point, then $\nu(P)$ is in the equator of S^2 for any $P \in \Sigma$. Hence the Gauss curvature K of Σ vanishes at any point. Since $K \equiv 0 \equiv H$, Σ is a plane which is parallel to the x_3-axis. □

5. Examples

Let $\gamma\colon \Omega \to \mathbf{R}_+$ be an axisymmetric positive C^∞ function (say, $\gamma(\nu) = \gamma(\nu_3)$) on a nonempty open set Ω in S^2. Assume that the matrix $D^2\gamma + \gamma 1$ is non-singular at each point $\nu \in \Omega$.

In this section, we study a special type of cyclic surfaces, that is, surfaces foliated by circles in parallel planes which are orthogonal to the x_3-axis. So our surfaces are represented as follows:

$$X(\theta, t) = (r(t)\cos\theta + f(t), r(t)\sin\theta + g(t), t). \tag{15}$$

As in Lemma 2, we denote by μ_1, μ_2 the principal curvatures of the standard body M_γ with respect to the normal $-\nu$, here μ_1 is the curvature of the generating curve of M_γ.

Lemma 4. *The anisotropic mean curvature of X in* (15) *is given by*

$$\Lambda = \frac{r(r'' + f''\cos\theta + g''\sin\theta) - (f'\sin\theta - g'\cos\theta)^2}{\mu_1\, r\{(r' + f'\cos\theta + g'\sin\theta)^2 + 1\}^{\frac{3}{2}}} - \frac{1}{\mu_2\, r\sqrt{(r' + f'\cos\theta + g'\sin\theta)^2 + 1}}. \tag{16}$$

Proof. Let ν be the Gauss map of X as usual. Let $\{e_1, e_2\}$ be a locally defined frame on S^2 such that $(D^2\gamma + \gamma 1)e_i = (1/\mu_i)e_i$. Note that the basis $\{e_1, e_2\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent plane of X at p. As in §2, let $(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2\gamma + \gamma 1)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & -w_{12}/\mu_1 \\ -w_{21}/\mu_2 & -w_{22}/\mu_2 \end{pmatrix},$$

and

$$\Lambda = w_{11}/\mu_1 + w_{22}/\mu_2 \tag{17}$$

holds. So, we will compute the matrix (w_{ij}).

Let $\nu^M = (\nu_1^M, \nu_2^M, \nu_3^M)$ be the outward pointing unit normal to M_γ. Since M_γ is a surface of revolution, $D^2\gamma + \gamma 1$ has eigendirections corresponding to

$$E_1 = (0, 0, 1) - \nu_3^M \nu^M, \quad E_2 = \nu^M \times E_1 \tag{18}$$

as long as the normal is not vertical. E_1, E_2 define an orthonormal basis $\{e_1, e_2\}$ on TS^2 as long as X does not intersect with the vertical axis.

Set $g_{11} = \langle X_\theta, X_\theta\rangle$, $g_{12} = g_{21} = \langle X_\theta, X_t\rangle$, $g_{22} = \langle X_t, X_t\rangle$, $h_{11} = \langle X_{\theta\theta}, \nu\rangle$, $h_{12} = h_{21} = \langle X_{\theta t}, \nu\rangle$, $h_{22} = \langle X_{tt}, \nu\rangle$. And set

$$\Delta := r' + f'\cos\theta + g'\sin\theta.$$

Then,

$$\begin{aligned}
g_{11} &= r^2, \\
g_{12} &= -rf'\sin\theta + rg'\cos\theta, \\
g_{22} &= (r')^2 + 2r'f'\cos\theta + 2r'g'\sin\theta + (f')^2 + (g')^2 + 1, \\
h_{11} &= \frac{-r}{\sqrt{\Delta^2 + 1}}, \\
h_{12} &= 0, \\
h_{22} &= \frac{r'' + f''\cos\theta + g''\sin\theta}{\sqrt{\Delta^2 + 1}},
\end{aligned}$$

and

$$\nu = (\nu_1, \nu_2, \nu_3) := \frac{X_\theta \times X_t}{|X_\theta \times X_t|} = \frac{1}{\sqrt{\Delta^2 + 1}}(\cos\theta, \sin\theta, -\Delta).$$

We have

$$\tilde{E}_1 := (0, 0, 1) - \nu_3\nu = \frac{1}{\Delta^2 + 1}(\Delta\cos\theta, \Delta\sin\theta, 1).$$

Hence,

$$e_1 = \frac{\tilde{E}_1}{|\ \tilde{E}_1\ |} = \frac{1}{\sqrt{\Delta^2+1}}\big(\Delta\cos\theta, \Delta\sin\theta, 1\big), \qquad (19)$$

$$e_2 = \nu \times e_1 = (\sin\theta, -\cos\theta, 0). \qquad (20)$$

Now we take a coordinate transformation $\theta(u,v)$, $t(u,v)$ so that, at an arbitrary fixed point (u_0, v_0),

$$\frac{\partial X}{\partial u} = e_1, \quad \frac{\partial X}{\partial v} = e_2$$

are satisfied. Then, we have

$$\frac{\partial X}{\partial \theta}\theta_u + \frac{\partial X}{\partial t}t_u = e_1, \quad \frac{\partial X}{\partial \theta}\theta_v + \frac{\partial X}{\partial t}t_v = e_2. \qquad (21)$$

Inserting $X_\theta = (-r\sin\theta, r\cos\theta, 0)$, $X_t = (r'\cos\theta + f', r'\sin\theta + g', 1)$, (19), and (20) to (21), we obtain

$$J := \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} = \begin{pmatrix} \frac{f'\sin\theta - g'\cos\theta}{r\sqrt{\Delta^2+1}} & -\frac{1}{r} \\ \frac{1}{\sqrt{\Delta^2+1}} & 0 \end{pmatrix}, \quad \det J > 0.$$

Let (w_{ij}), $(\tilde{w}_{ij})$ be the Weingarten mappings for $X(u,v)$, $X(\theta,t)$, respectively. Then,

$$(\tilde{w}_{ij}) = (g_{ij})^{-1}(h_{ij}),$$

$$(w_{ij}) = \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix}^{-1} (\tilde{w}_{ij}) \begin{pmatrix} \theta_u & \theta_v \\ t_u & t_v \end{pmatrix} = J^{-1}(\tilde{w}_{ij})J.$$

Hence, by a computation, we obtain

$$w_{11} = \frac{r(r'' + f''\cos\theta + g''\sin\theta) - (f'\sin\theta - g'\cos\theta)^2}{r(\Delta^2+1)^{\frac{3}{2}}},$$

$$w_{22} = -\frac{1}{r\sqrt{\Delta^2+1}}.$$

This with (17) gives (16). □

Now we assume that the standard body M_γ for γ is a quadric surface of revolution. Then, by homothety and translation, M_γ is one of the followings:

(I) a spheroid: $x_1^2 + x_2^2 + \dfrac{x_3^2}{a^2} = 1$,

(II) a hyperboloid of two sheets: $x_1^2 + x_2^2 - \dfrac{x_3^2}{a^2} = -1$,

(III) a hyperboloid of one sheet: $x_1^2 + x_2^2 - \dfrac{x_3^2}{a^2} = 1$,

(IV) a circular paraboloid: $x_3 = a(x_1^2 + x_2^2)$,

where a is a positive constant.

Lemma 5. *The support functions γ of M_γ in the above (I)–(IV) are respectively given by the followings:*

(I) $\gamma(\nu_3) = \sqrt{1 + b\nu_3^2}$, $(b := a^2 - 1 > -1)$,

(II) $\gamma(\nu_3) = \sqrt{-1 + b\nu_3^2}$, $(b := a^2 + 1 > 1,\ \frac{1}{\sqrt{b}} < |\nu_3| \leq 1)$,

(III) $\gamma(\nu_3) = \sqrt{1 - b\nu_3^2}$, $(b := a^2 + 1 > 1,\ |\nu_3| < \frac{1}{\sqrt{b}})$,

(IV) $\gamma(\nu_3) = \dfrac{-1 + \nu_3^2}{b\nu_3}$, $(b := 4a > 0,\ \nu_3 \neq 0)$.

Proof. (I) Represent the upper half of M_γ as

$$Y(x_1, x_2) = \Big(x_1, x_2, a\sqrt{1 - x_1^2 - x_2^2}\Big).$$

The outward pointing unit normal ν to Y is given by

$$\nu = \frac{1}{\sqrt{1 + (a^2-1)(x_1^2 + x_2^2)}}\Big(ax_1, ax_2, \sqrt{1 - x_1^2 - x_2^2}\Big).$$

Hence, we obtain

$$\gamma = \langle Y, \nu\rangle = \sqrt{1 + (a^2-1)\nu_3^2} = \sqrt{1 + b\nu_3^2},$$

which proves (I).

Similarly, we obtain (II)–(IV). □

Proposition 3. *Let γ be a function given by the above (I)–(IV). Then, there exist γ-minimal surfaces foliated by circles contained in parallel planes which are orthogonal to the x_3-axis. Up to translations in $\mathbf{R}^3$, rotations around the x_3-axis, and symmetry with respect to a plane $\{x_3 =$ constant$\}$, they are respectively represented as follows.*

(I) Catenoid-type:

$$X(\theta,t) = \left(\frac{\cosh(ct)}{c\sqrt{1+b}}\cos\theta, \frac{\cosh(ct)}{c\sqrt{1+b}}\sin\theta, t\right), \quad c \neq 0. \qquad (22)$$

Riemann-type:

$$X(\theta,r) = \Bigg(r\cos\theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 - 1}}, r\sin\theta, \sqrt{1+b}\int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 - 1}}\Bigg), \qquad (23)$$

$$c_1 \neq 0,\ r \geq \left(\frac{-c_2 + \sqrt{c_2^2 + 4c_1^2}}{2c_1^2}\right)^{1/2}. \qquad (24)$$

(II) Catenoid-type:

$$X(\theta,t) = \left(\frac{\sinh(ct)}{c\sqrt{b-1}}\cos\theta, \frac{\sinh(ct)}{c\sqrt{b-1}}\sin\theta, t\right), \qquad (25)$$

$$c \neq 0,\ t \neq 0. \qquad (26)$$

Riemann-type:

$$X(\theta,r) = \Bigg(r\cos\theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r\sin\theta, \sqrt{b-1}\int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}\Bigg), \qquad (27)$$

$$c_1 \neq 0,\ c_2 > 2|c_1|,\ r > 0. \qquad (28)$$

(III) Catenoid-type:

$$X(\theta,t) = \left(\frac{\sin(ct)}{c\sqrt{b-1}}\cos\theta, \frac{\sin(ct)}{c\sqrt{b-1}}\sin\theta, t\right), \qquad (29)$$

$$c \neq 0,\ \sin(ct) \neq 0. \qquad (30)$$

Riemann-type:

$$X(\theta, r) = \left(r\cos\theta + \int \frac{c_1 r^2 dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}}, r\sin\theta, \right.$$
$$\left. \sqrt{b-1} \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2 + 1}} \right), \qquad (31)$$

$$c_1 \neq 0, c_2 < -2|c_1|, 0 < r \leq \left(\frac{|c_2| - \sqrt{c_2^2 - 4c_1^2}}{2c_1^2} \right)^{1/2}. \qquad (32)$$

(IV) Catenoid-type:

$$X(\theta, t) = (e^{ct}\cos\theta, e^{ct}\sin\theta, t), \quad c \neq 0. \qquad (33)$$

Riemann-type:

$$X(\theta, r) = \left(r\cos\theta + \int \frac{c_1 r dr}{\sqrt{c_1^2 r^2 + c_2}}, \right.$$
$$\left. r\sin\theta, \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2}} \right), \qquad (34)$$
$$c_1 \neq 0, \ c_2 > 0, \ r > 0. \qquad (35)$$

(I) Catenoid-type (I) Riemann-type

Figure 1: γ-minimal surfaces for $\gamma(\nu)$ as in (I) of Lemma 5.

(II) Catenoid-type (II) Riemann-type

Figure 2: γ-minimal surfaces for $\gamma(\nu)$ as in (II) of Lemma 5.

(III) Catenoid-type (III) Riemann-type

Figure 3: γ-minimal surfaces for $\gamma(\nu)$ as in (III) of Lemma 5.

Remark 1. Let γ be a function given in (I), (II), or (III) in Lemma 5. Set $\alpha := \sqrt{b+1}$ for (I), and $\alpha := \sqrt{b-1}$ for (II) and (III). If we take the transformation $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = x_3/\alpha$, then an immersion $X = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ is γ-minimal if and only if $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a minimal

(IV) Catenoid-type (IV) Riemann-type

Figure 4: γ-minimal surfaces for $\gamma(\nu)$ as in (IV) of Lemma 5.

surface (for (I)), a spacelike zero mean curvature surface in $\mathbf{R}^3_1$ (for (II)), a timelike zero mean curvature surface in $\mathbf{R}^3_1$ (for (III)), respectively. This is proved by the same way as Example 4.4 in [8].

Remark 2. γ-minimal surfaces for γ given in (IV) in Lemma 5 are graphs of harmonic functions (cf. [12]).

Proof of Proposition 3. Let μ_1, μ_2 be the principal curvatures of M_γ with respect to the normal $-\nu$ in the same way as Lemma 2.

We represent the surface as

$$X(\theta, t) = (r(t)\cos\theta + f(t), r(t)\sin\theta + g(t), t). \qquad (36)$$

Note that X is a surface of revolution around the x_3-axis if and only if $f \equiv g \equiv 0$ holds.

As in the proof of Lemma 4, we set

$$\Delta := r' + f'\cos\theta + g'\sin\theta.$$

Then, the Gauss map ν of X is

$$\nu(\theta, t) := \frac{X_\theta \times X_t}{|X_\theta \times X_t|}$$
$$= \frac{1}{(\Delta^2 + 1)^{1/2}} (\cos\theta, \sin\theta, -\Delta).$$

(I) By a simple computation using Lemma 2, we obtain

$$\frac{1}{\mu_1} = \frac{b+1}{(1 + b\nu_3^2)^{3/2}}, \quad \frac{1}{\mu_2} = \frac{1}{\sqrt{1 + b\nu_3^2}}.$$

Since $\nu_3 = \dfrac{-\Delta}{(\Delta^2+1)^{1/2}}$, we obtain

$$\frac{1}{\mu_1} = (b+1) \left(\frac{1 + \Delta^2}{1 + (b+1)\Delta^2} \right)^{3/2}, \qquad (37)$$

$$\frac{1}{\mu_2} = \left(\frac{1 + \Delta^2}{1 + (b+1)\Delta^2} \right)^{1/2}. \qquad (38)$$

By Lemma 4 with (37) and (38), we see that $\Lambda = 0$ if and only if

$$(b+1)(rf'' - 2r'f')\cos\theta + (b+1)(rg'' - 2r'g')\sin\theta$$
$$+ (b+1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \qquad (39)$$
$$rg'' - 2r'g' = 0, \qquad (40)$$
$$(b+1)(r''r - (r')^2 - (f')^2 - (g')^2) - 1 = 0. \qquad (41)$$

From (39) and (40), we have

$$f' = c_1 r^2, \quad g' = c_2 r^2. \tag{42}$$

First, assume $f' = g' = 0$. Then, (41) is equivalent to

$$(r''r - (r')^2) - \frac{1}{b+1} = 0. \tag{43}$$

By a standard way, we see that the general solution of (43) is

$$r = \frac{\cosh(c_3(t + c_4))}{c_3\sqrt{b+1}}, \quad c_3 \neq 0, \tag{44}$$

which gives the formula (22).

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. Because of (42), $c_2 f' - c_1 g' = 0$ holds. This implies that $c_2 f - c_1 g =$ constant. Since $(c_1, c_2) \neq (0, 0)$, by rotating the surface around the x_3-axis if necessary, we may assume that

$$f' = c_1 r^2 \ (c_1 \neq 0), \quad g(t) \equiv 0 \tag{45}$$

holds. Then, (41) is equivalent to

$$1 + (b+1)((r')^2 - r''r + c_1^2 r^4) = 0.$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm\sqrt{c_1^2 r^4 + c_5 r^2 - \frac{1}{b+1}}\,. \tag{46}$$

Hence,

$$\begin{aligned} t &= \pm \int \frac{dr}{\sqrt{c_1^2 r^4 + c_5 r^2 - \frac{1}{b+1}}} \\ &= \pm\sqrt{b+1} \int \frac{dr}{\sqrt{c_6^2 r^4 + c_7 r^2 - 1}}, \\ &\qquad c_6 := \sqrt{b+1}c_1 \neq 0, \quad c_7 := (b+1)c_5. \end{aligned} \tag{47}$$

By using (45) and (46), we easily obtain

$$f = \pm \int \frac{c_6 r^2 \, dr}{\sqrt{c_6^2 r^4 + c_7 r^2 - 1}}, \quad c_6 \neq 0. \tag{48}$$

(47) with (48) gives the formula (23). Moreover,

$$c_1^2 r^4 + c_2 r^2 - 1 \geq 0$$

if and only if

$$r \geq \left(\frac{-c_2 + \sqrt{c_2^2 + 4c_1^2}}{2c_1^2} \right)^{1/2}$$

holds, which gives the condition (24).

(II) The proof is similar to the proof of (I). We have

$$\begin{aligned} \frac{1}{\mu_1} &= \frac{-(b-1)}{(-1 + b\nu_3^2)^{3/2}} \\ &= -(b-1)\left(\frac{1 + \Delta^2}{-1 + (b-1)\Delta^2} \right)^{3/2}, \end{aligned}$$

$$\frac{1}{\mu_2} = \frac{-1}{\sqrt{-1 + b\nu_3^2}} = -\left(\frac{1 + \Delta^2}{-1 + (b-1)\Delta^2} \right)^{1/2}.$$

We see that $\Lambda = 0$ if and only if

$$\begin{aligned} &(b-1)(rf'' - 2r'f')\cos\theta + (b-1)(rg'' - 2r'g')\sin\theta \\ &\quad + (b-1)\{r''r - (r')^2 - (f')^2 - (g')^2\} + 1 = 0 \end{aligned}$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \tag{49}$$

$$rg'' - 2r'g' = 0, \tag{50}$$

$$(b-1)(r''r - (r')^2 - (f')^2 - (g')^2) + 1 = 0. \tag{51}$$

First, assume $f' = g' = 0$. Then, (51) is equivalent to

$$(r''r - (r')^2) + \frac{1}{b-1} = 0. \tag{52}$$

We have the following three types of general solutions of (52):

$$r = \frac{1}{\sqrt{b-1}} t + c, \tag{53}$$

$$r = \frac{\sinh(c_1(t + c_2))}{c_1\sqrt{b-1}}, \quad c_1 \neq 0, \tag{54}$$

$$r = \frac{\sin(c_1(t + c_2))}{c_1\sqrt{b-1}}, \quad c_1 \neq 0. \tag{55}$$

By a suitable translation, the corresponding surfaces are given by

$$X(\theta, t) = \left(\frac{t}{\sqrt{b-1}}\cos\theta, \frac{t}{\sqrt{b-1}}\sin\theta, t \right), \tag{56}$$

$$X(\theta, t) = \left(\frac{\sinh(ct)}{c\sqrt{b-1}}\cos\theta, \frac{\sinh(ct)}{c\sqrt{b-1}}\sin\theta, t \right), \ c \neq 0, \tag{57}$$

$$X(\theta, t) = \left(\frac{\sin(ct)}{c\sqrt{b-1}}\cos\theta, \frac{\sin(ct)}{c\sqrt{b-1}}\sin\theta, t \right), \ c \neq 0, \tag{58}$$

respectively. Later, we will show that in order that the surface is compatible with γ, the surface must be given by (57), which gives the formula (25).

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. By rotating the surface around the x_3-axis if necessary, we may assume that

$$f' = c_1 r^2 \ (c_1 \neq 0), \quad g(t) \equiv 0 \tag{59}$$

holds. Then, (51) is equivalent to

$$1 - (b-1)((r')^2 - r''r + c_1^2 r^4) = 0. \tag{60}$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm\sqrt{c_1^2 r^4 + c_2 r^2 + \frac{1}{b-1}}\,. \tag{61}$$

Hence,

$$\begin{aligned} t &= \pm\sqrt{b-1} \int \frac{dr}{\sqrt{c_3^2 r^4 + c_4 r^2 + 1}}, \\ &\qquad c_3 := \sqrt{b-1}c_1 \neq 0, \ c_4 := (b-1)c_2. \end{aligned} \tag{62}$$

By using $f' = c_1 r^2$ and (61), we obtain

$$f = \pm \int \frac{c_3 r^2\, dr}{\sqrt{c_3^2 r^4 + c_4 r^2 + 1}}, \quad c_3 \neq 0. \tag{63}$$

(62) with (63) gives the formula (27).

Next, we will check whether the surface is compatible with γ or not. Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $b\nu_3^2 - 1 > 0$ for all θ.

For the surface X given by (56), the Gauss map ν is

$$\begin{aligned}\nu &:= \frac{X_\theta \times X_t}{|X_\theta \times X_t|} \\ &= b^{-1/2}\Big(\sqrt{b-1}\cos\theta, \sqrt{b-1}\sin\theta, -1\Big).\end{aligned}$$

This shows that $b\nu_3^2 - 1 \equiv 0$, and hence X is not compatible with γ.

For the surface X given by (57),

$$\nu = \Big(1 + \frac{\cosh^2(ct)}{b-1}\Big)^{-1/2}\Big(\cos\theta, \sin\theta, -\frac{\cosh(ct)}{\sqrt{b-1}}\Big).$$

This shows that $b\nu_3^2 - 1 \geq 0$ always holds, and that $b\nu_3^2 - 1 > 0$ for all θ if and only if $t \neq 0$.

For the surface X given by (58),

$$\nu = \Big(1 + \frac{\cos^2(ct)}{b-1}\Big)^{-1/2}\Big(\cos\theta, \sin\theta, -\frac{\cos(ct)}{\sqrt{b-1}}\Big). \tag{64}$$

This shows that $b\nu_3^2 - 1 \leq 0$ always holds, and hence X is not compatible with γ.

For the surface X given by (27), the Gauss map ν is

$$\nu := \frac{X_\theta \times X_r}{|X_\theta \times X_r|}, \tag{65}$$

$$\begin{aligned}X_\theta \times X_r = \Bigg(&\frac{\sqrt{b-1}r\cos\theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}}, \frac{\sqrt{b-1}r\sin\theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}}, \\ &-r - \frac{c_1 r^3 \cos\theta}{(c_1^2 r^4 + c_2 r^2 + 1)^{1/2}}\Bigg).\end{aligned} \tag{66}$$

This shows that, $b\nu_3^2 - 1 > 0$ for all θ if and only if $c_2 > 2|c_1|$ holds.

(III) The proof is again similar to the proof of (I). We see that the condition $\Lambda = 0$ is equivalent to the condition that the system of ordinary differential equations (49), (50) and (51) holds.

First, assume $f' = g' = 0$. Then, (51) is equivalent to (52). Note that $b - 1 > 0$. The general solutions of (52) are given by (53), (54), and (55), and corresponding surfaces are given by (56), (57), and (58). Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $1 - b\nu_3^2 > 0$ for all θ. As we have seen above, for the surfaces (56) and (57), $b\nu_3^2 - 1 \geq 0$ holds at every point. Hence, they are not compatible with γ. Therefore, only the possibility is the case (58), which is the same as the formula (29). The Gauss map ν for this surface is given by (64), which shows that, $1 - b\nu_3^2 > 0$ for all θ if and only if $\sin(ct) \neq 0$.

Next, we assume that $f' \neq 0$ or $g' \neq 0$ holds. By rotating the surface around the x_3-axis if necessary, we may assume that f and g satisfy (59). Then, (51) is equivalent to (60). Hence, we obtain the formula (31).

We will check whether the surface is compatible with γ or not. The Gauss map ν is given by (65), (66) as in the case (II). This shows that, $1 - b\nu_3^2 > 0$ for all θ if and only if both $c_2 < -2|c_1|$ and

$$0 < r \leq \left(\frac{|c_2| - \sqrt{c_2^2 - 4c_1^2}}{2c_1^2}\right)^{1/2}$$

hold, which gives the condition (32).

(IV) The proof is again similar to the proof of (I). We obtain

$$\frac{1}{\mu_1} = \frac{-2}{b\nu_3^3} = \frac{2(\Delta^2+1)^{3/2}}{b\Delta^3}, \tag{67}$$

$$\frac{1}{\mu_2} = \frac{-2}{b\nu_3} = \frac{2(\Delta^2+1)^{1/2}}{b\Delta}. \tag{68}$$

By Lemma 4 with (67) and (68), we see that $\Lambda = 0$ if and only if

$$\begin{aligned}&(rf'' - 2r'f')\cos\theta + (rg'' - 2r'g')\sin\theta \\ &\qquad + (r''r - (r')^2 - (f')^2 - (g')^2) = 0\end{aligned}$$

holds. This gives the following system of ordinary differential equations:

$$rf'' - 2r'f' = 0, \tag{69}$$

$$rg'' - 2r'g' = 0, \tag{70}$$

$$r''r - (r')^2 - (f')^2 - (g')^2 = 0. \tag{71}$$

From (69) and (70), we have

$$f' = c_1 r^2, \quad g' = c_2 r^2. \tag{72}$$

When $f' = g' = 0$ holds, (71) is equivalent to

$$r''r - (r')^2 = 0. \tag{73}$$

The general solution of (73) is

$$r = e^{c_1 t + c_2},$$

which gives the formula (33).

When $f' \neq 0$ or $g' \neq 0$ holds, by rotating the surface around the x_3-axis if necessary, we may assume that

$$f' = c_1 r^2 \ (c_1 \neq 0), \quad g(t) \equiv 0 \tag{74}$$

holds. Then, (71) is equivalent to

$$(r')^2 - r''r + c_1^2 r^4 = 0. \tag{75}$$

From this, by a standard argument, we obtain

$$\frac{dr}{dt} = \pm\sqrt{c_1^2 r^4 + c_2 r^2}. \tag{76}$$

Hence,

$$t = \pm \int \frac{dr}{\sqrt{c_1^2 r^4 + c_2 r^2}}. \qquad (77)$$

By using $f' = c_1 r^2$ and (76), we obtain

$$f = \pm \int \frac{c_1 r \, dr}{\sqrt{c_1^2 r^2 + c_2}}, \quad c_1 \neq 0. \qquad (78)$$

(77) with (78) gives the formula (34).

Next, we will check whether the surface is compatible with γ or not. Note that the surface X is compatible with γ if and only if its Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies $\nu_3 \neq 0$ for all θ.

For the surface X given by (33), the Gauss map ν is

$$\begin{aligned} \nu &:= \frac{X_\theta \times X_t}{|X_\theta \times X_t|} \\ &= \left(1 + c^2 e^{2ct}\right)^{-1/2} \left(\cos\theta, \sin\theta, -ce^{ct}\right). \end{aligned}$$

This shows that, $\nu_3 \neq 0$ for all θ if and only if $c \neq 0$.

For the surface X given by (34), the Gauss map ν is

$$\nu := \frac{X_\theta \times X_r}{|X_\theta \times X_r|},$$

$$X_\theta \times X_r = \left(\frac{\cos\theta}{(c_1^2 r^2 + c_2)^{1/2}}, \frac{\sin\theta}{(c_1^2 r^2 + c_2)^{1/2}}, - r\left(1 + \frac{c_1 r \cos\theta}{(c_1^2 r^2 + c_2)^{1/2}}\right) \right).$$

This shows that, $\nu_3 \neq 0$ for all θ if and only if $c_2 > 0$ holds. □

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References

[1] Andrews, B.: Volume-preserving anisotropic mean curvature flow, *Indiana Univ. Math. J.* **50** (2001), 783–827.

[2] Fujimori, S., Kim, Y. W., Koh, S.-E., Rossman, W., Shin, H., Umehara, M., Yamada, K., and Yang, S.-D.: Zero mean curvature surfaces in Lorentz-Minkowski 3-space which change type across a light-like line, Preprint.

[3] Gu, C.: The extremal surfaces in the 3-dimensional Minkowski space, *Acta. Math. Sinica* **1** (1985), 173–180.

[4] Kim, Y. W. , Koh, S.-E., Shin H., and Yang, S.-D.: Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae, *Journal of Korean Math. Soc.* **48** (2011), 1083–1100.

[5] Kim, Y. W. and Yang, S.-D.: Prescribing singularities of maximal surfaces via a singular Björling representation formula, *J. Geom. Phys.* **57** (2007), 2167–2177.

[6] Klyachin, V. A.: Zero mean curvature surfaces of mixed type in Minkowski space, *Izv. Math.* **67** (2003), 209–224.

[7] Kobayashi, O.: Maximal surfaces in the 3-dimensional Minkowski space L^3, *Tohoku J. Math.* **6** (1983), 297–309.

[8] Koiso, M. and Palmer, B.: Geometry and stability of surfaces with constant anisotropic mean curvature, *Indiana Univ. Math. J.* **54** (2005), 1817–1852.

[9] Kuhns, C. and Palmer, B.: Helicoidal surfaces with constant anisotropic mean curvature, *J. Math. Physics* **52** (2011), 073506-1–14.

[10] López, F., López, R. and Souam, R.: Maximal surfaces of Riemann type in Lorentz-Minkowski space L^3, *Michigan Math. J.* **47** (2000), 469–497.

[11] López, R.: Timelike surfaces with constant mean curvature in Lorentz three-space, *Tohoku J. Math.* **52** (2000), 515–532.

[12] Reilly, R. C.: The relative differential geometry of nonparametric hypersurfaces, *Duke Math. J.* **43** (1976), 705–721.

[13] Taylor, J. E.: Crystalline variational problems, *Bull. Amer. Math. Soc.* **84** (1978), 568–588.

[14] Winterbottom, W. L.: Equilibrium shape of a small particle in contact with a foreign substrate, *Acta Metallurgica* **15** (1967), 303–310.

[15] Wulff, G.: Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflächen, *Zeitschrift für Krystallographie und Mineralogie* **34** (1901), 449–530.

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