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L^2-theoretical study of the relation between the LIBOR market model and the HJM model

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Abstract. In previous works, the author introduced metric spaces of term structure models to study the relation between the LIBOR market model and the HJM model. However that framework is not comprehensive, nor does it admit an extendable structure.
This paper introduces a new metric space to better develop the perspective argument. A metric space is naturally constructed on the set of bond price processes such that the space allows many types of term structure models. This metric presents a general view on the relation between the LIBOR market model and the HJM model. Consequently, the LIBOR market model is placed at the boundary of the HJM model set.

Keywords. term structure model, LIBOR market model, BGM model, HJM model, metric space

1. Introduction

There are two streams of thought related to the term structures of interest rate modeling. One is the framework of Heath et al. [4] (hereafter HJM), which is based on arbitrage-free dynamics of the instantaneous forward rates. Another, called the LIBOR market model, is based on forward LIBOR rates, and has been studied by Miltersen et al. [7], Brace et al. [1] (hereafter BGM), Jamshidian [6] and Musiela and Rutkowski [8]. This model is most popular among practitioners, because it admits caplet formulas and swaption approximation formulas. However the BGM model requires smoothness of volatility, because it is constructed within the framework of HJM. On the other hand, [8] and [6] construct models similar to BGM in a general manner, based on the discrete family of bond price processes without use of the instantaneous forward rate. Then this discrete framework is not based on the HJM framework, nor does it require volatility smoothness.

With respect to the volatility smoothness, the author's previous work (Yasuoka [10]) shows that even if BGM model volatility does not satisfy smoothness, the corresponding LIBOR process is arbitrarily approximated by the BGM model with smooth volatility. [10] considered the space of volatilities to explain topologically the relation between the BGM model with smooth volatility and that without smoothness. A metric was furthermore introduced on this volatility space. In this setting, [10] shows that the BGM model without smoothness is situated on the boundary of the set of the BGM model with smoothness. However this volatility space involves only the BGM model, and the definition of the metric is slightly ambiguous. This investigation should therefore be reconstructed more comprehensively.

The aim of this paper is to generalize the argument presented in [10]. Section 2 constructs another space, one that includes many types of term structure model, including the HJM model and the discrete family of bonds. Another metric is also introduced on this space, one that is simpler than the previous one.

In Section 3, we briefly introduce the term structure models, HJM, BGM, and the LIBOR market model. To avoid confusion, the BGM model with smoothness is classified into the HJM model. In a strict sense, the LIBOR market model is referenced as a model introduced by [8] and [6], which is not based on the HJM framework. Hence the BGM model without smoothness is classified into the LIBOR market model.

Finally, we shall show that the LIBOR market model (or the BGM model without smoothness) is placed at the boundary[1] of the HJM model set. As a result, the new metric presents a perspective view on the relation among term structure models.

2. Metric space of bond price processes

Let $\delta > 0$ be a positive constant, and a positive integer n be fixed. A sequence of time $\{T_i\}$ is defined as $T_i = \delta i$, $i = 1, 2, \ldots, n$. Furthermore let $(\Omega, \mathcal{F}, \mathsf{P}, \{\mathcal{F}_t\}_{t\in[0,T_n]})$ be a filtered probability space, where $\{\mathcal{F}_t\}_{t\in[0,T_n]}$ is the augmented filtration, and P is called the original measure (or the real-world measure). We set

$$J = \{(i,j) \in \mathbf{N}^2 : 0 \le i, j \le n,\ i \le j\},$$
$$K = \{(i,k) \in \mathbf{N}^2 : 0 \le i, k \le n,\ i + k \le n\}.$$

[1] cf. the footnote in Section 4.

Let $P(t,T)$ be an $\mathcal{F}_t$-adapted process, sometimes denoted by P for simplicity. $\mathcal{X}$ denotes a set of $P(t,T)$ defined by

$$\mathcal{X} = \left\{ P(t,T) : \max_{(i,j)\in J} E[P(T_i,T_j)^2] < \infty \right\},$$

where $E[\,]$ denotes the expectation under P.

We introduce an equivalence relation R on $\mathcal{X}$ as follows. For P_1 and P_2 in $\mathcal{X}$, P_1 is said to be *equivalent* to P_2 if and only if it holds that

$$P_1(T_i,T_j) = P_2(T_i.T_j) \quad a.s.\mathsf{P}$$

for all $(i,j)\in J$. The quotient space $\mathcal{X}/R$ is denoted by $\mathcal{Y}$. Without loss of generality, we may consider that P always expresses the equivalence class with respect to this relation. Accordingly, a metric on $\mathcal{Y}$ is defined by

$$d(P_1,P_2) = \max_{(i,j)\in J} \left\{ E[(P_1(T_i,T_j) - P_2(T_i,T_j))^2] \right\}^{1/2} \quad (1)$$

for $P_1, P_2 \in \mathcal{Y}$. Obviously, $\mathcal{Y}$ is a complete metric space[2]. Furthermore, for a measure Q equivalent to P, $\mathcal{X}_\mathcal{Q}$ and $\mathcal{Y}_\mathcal{Q}$ are similarly defined as above. Naturally, $\mathcal{X}_\mathcal{Q}$ is not always equivalent to $\mathcal{X}$, nor $\mathcal{Y}_\mathcal{Q}$ to $\mathcal{Y}$.

Next, let $B = (B_1(t),\ldots,B_n(t))$ be a discrete family of adapted processes such that

$$\max_{(i,j)\in J} E[B_j(T_i)^2] < \infty.$$

Consider $P(t,T)$ in $\mathcal{Y}$ such that it satisfies

$$P(t,T_j) = B_j(t)$$

for all j and $t \geq 0$. $P(t,T)$ is uniquely determined in the quotient space $\mathcal{Y}$. Identifying B with $P(t,T)$, we may consider that $B \in \mathcal{Y}$.

We are interested in the case where $P(t,T)$ is a bond price process with maturity T. Usually, the expiry date of LIBOR and swap derivatives is set at some maturity T_i. We denote LIBOR observed at T_i over the period $[T_{i+l}, T_{i+l+1}]$ by $L(T_i,T_l)$, which is given by

$$1 + \delta L(T_i,T_l) = \frac{P(T_i,T_{i+l+1})}{P(T_i,T_{i+l})}$$

for $l = 0,\ldots,n-i-1$. The swap rate at T_i is expressed by a function of $L(T_i,T_l)$, and then expressed by a function of $P(T_i,T_{i+l})$. Then the prices of LIBOR and swap derivatives are generally determined by $P(T_i,T_j)$, $j \leq i$, that is, any interpolation of $P(T_i,T_j)$ does not affect the derivative price[3]. Hence, the quotient space $\mathcal{Y}$ is well-defined in the sense of derivative pricing. Note that the difference with $\mathcal{Y}$ from the previous space defined in [10] is remarked upon at the end of the next section.

Particularly, for a bond price process $P(t,T)$ and a measure Q, if there exists a numeraire asset whose price is denoted by $N(t)$ such that $P(t,T)/N(t)$ is a Q-martingale, then $P(t,T)$ is arbitrage-free. Regarding $P(t,T)$ as an element in $\mathcal{Y}_\mathcal{Q}$, we denote the set of all arbitrage-free bond price processes by $\mathcal{N}_Q$. For a discrete family of bond price processes $B = (B_1(t),\ldots,B_n(t))$, if there exists a numeraire $N(t)$ such that $B_i(t)/N(t)$, $i = 1,\ldots,n$ are Q-martingales, B is arbitrage-free. In this case, we may consider that B is an element in $\mathcal{N}_Q$.

3. Term structure within the HJM framework

First we introduce the framework of the HJM model. $f(t,T)$ denotes the instantaneous forward rate (hereinafter, the forward rate) with maturity $T \leq T_n$ prevailing at time $t \leq T$. The dynamics of $f(t,T)$ are assumed to be expressed by

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dZ_t, \quad (2)$$

where Z_t is an $\mathbf{R}^d$-valued Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P}, \{\mathcal{F}_t\}_{t\in[0,T_n]})$ (hereinafter, P Brownian motion), and $\alpha(t,T)$ and $\sigma(t,T)$ are $\mathbf{R}^d$-valued adapted processes. The instantaneous spot rate (hereinafter, the spot rate) is given by

$$s(t) = f(t,t).$$

There exists a measure Q equivalent to P such that $f(t,T)$ is described by

$$df(t,T) = \left(\sigma(t,T)\cdot \int_t^T \sigma(t,u)\,du \right) dt + \sigma(t,T)\cdot dW_t, \quad (3)$$

where W_t is an $\mathbf{R}^d$-valued Q Brownian motion. $P(t,T)$ denotes the price of the zero-coupon bond at time t with maturity T,

$$P(t,T) = \exp\left\{ -\int_t^T f(t,u)\,du \right\}.$$

The savings account $\beta(t)$ is defined by

$$\beta(t) = \exp\left\{ \int_0^t s(u)\,du \right\},$$

taken to be a numeraire. Thus $P(t,T)/\beta(t)$ is a Q-martingale for all T, and $P(t,T)$ is arbitrage-free. Note that the HJM framework takes the savings account as a numeraire. Then this model must require the existence of the forward rate process.

We denote the set of all bond price processes implied from the HJM framework by $\mathcal{H}_Q$. Then the inclusion relation among $\mathcal{H}_Q$, $\mathcal{N}_Q$ and $\mathcal{Y}_Q$ is as follows.

$$\mathcal{H}_Q \subset \mathcal{N}_Q \subset \mathcal{Y}_Q. \quad (4)$$

Next we sketch the BGM framework [1]. We set

$$r(t,x) = f(t,t+x).$$

[2]Here "complete" is used in the topological sense, not in the financial sense.

[3]This view was originally introduced in [6].

Let $\nu(t,x)$ be an $\mathbf{R}^d$-valued adapted volatility process. Consider the following equation:

$$dr(t,x) = \left\{\frac{\partial r(t,x)}{\partial x} + \nu(t,x)\cdot\nu_x(t,x)\right\}dt + \nu_x(t,x)\cdot dW_t, \tag{5}$$

where $\nu_x(t,x) = \partial\nu(t,x)/\partial x$. It is shown in [1] that if $\nu(t,x)$, $t \geq 0$ is adapted, and $\nu_x(t,x)$ is bounded, then there exists a unique mild solution[4] $r(t,x)$ to (5). $r(t,x)$ is represented by

$$\begin{aligned} r(t,x) = r(0,t+x) & \\ &+ \int_0^t \nu_x(s,x+t-s)\cdot\nu(s,x+t-s)\,ds \\ &+ \int_0^t \nu_x(s,x+t-s)\cdot dW_s. \end{aligned} \tag{6}$$

It follows that

$$\begin{aligned} \frac{P(t,T)}{\beta(t)} = P(0,T)\exp&\left\{-\int_0^t \nu(s,T-s)\cdot dW_s\right. \\ &\left. -\frac{1}{2}\int_0^t |\nu(s,T-s)|^2\,ds\right\} \end{aligned} \tag{7}$$

Therefore $P(t,T)/\beta(t)$ is a Q-martingale and then $P(t,T)$ is arbitrage-free. To verify the existence of $r(t,x)$, we give the following definition.

Definition 1. $\gamma(t,x)$ is said to be regular if for all $t \geq 0$, $\gamma(t,x)$ and

$$M(t,x) = \int_0^t \gamma(s,x+t-s)\cdot dW_s$$

is differentiable in $x \in \mathbf{R}^+$, moreover $\gamma(t,0) = 0$ and

$$\frac{\partial}{\partial x}\gamma(t,x)|_{x=0} = 0.$$

Let an initial rate $r(0,x)$ be positive and continuous in x. [1] shows that if $\gamma(t,x)$ is deterministic, bounded, piecewise continuous, and regular, then (5) has a unique solution $r(t,x)$ such that $r(t,x)$ is continuous in x.

Next, LIBOR at time t over the period $[t+x, t+x+\delta]$ is given by

$$1+\delta L(t,x) = \exp\left\{\int_{x+t}^{x+t+\delta} f(t,u)du\right\}$$

for $t \geq 0$. Obviously it holds that

$$1+\delta L(t,x) = \frac{P(t,t+x+\delta)}{P(t,t+x)}.$$

Suppose that the dynamics of $L(t,x)$ are given by

$$\begin{aligned} dL(t,x) = &\left\{\frac{\partial L(t,x)}{\partial x} + L(t,x)\gamma(t,x)\cdot\nu(t,x)\right. \\ &\left.+\frac{\delta L^2(t,x)}{1+\delta L(t,x)}|\gamma(t,x)|^2\right\}dt + L(t,x)\gamma(t,x)\cdot dW_t, \end{aligned} \tag{8}$$

where $\gamma(t,x)$ is an $\mathbf{R}^n$-valued volatility, and $\nu(t,x)$ is given by

$$\nu(t,x) = \sum_{i=1}^{[x/\delta]} \frac{\delta L(t,x-\delta i)}{1+\delta L(t,x-\delta i)}\gamma(t,x-\delta i). \tag{9}$$

Note that (9) implies $\nu(t,x) = 0$ for $0 \leq x < \delta$. From [1] if $\gamma(t,x)$ is deterministic, bounded and piecewise-continuous, and $L(0,x) > 0$, then there exists a unique solution $L(t,x) > 0$ to (8) for $t \geq 0$. Here, $P(t,T)$ is referred to as the *BGM model* if the LIBOR process is expressed by (8) with a regular volatility $\gamma(t,x)$.

Example 3.1. Every constant volatility $\gamma(t,x) \equiv a\ (\neq 0)$ is not regular, because $\gamma(0,x) \neq 0$. In this case, there exists a LIBOR process but no instantaneous forward rate process. This is a trivial example of a non-BGM model. □

When $\gamma(t,x)$ is not regular, the forward rate process is not obtained. Then the term structure dynamics are not analyzed in the HJM framework. However, if the LIBOR process exists, a discrete family of bond price processes is defined by an arbitrary adapted process B satisfying

$$B_j(T_i) = \prod_{l=0}^{j-1-i}(1+\delta L(T_i,T_l))^{-1} \tag{10}$$

for all $(i,j) \in J$. It holds that $B_j(T_j) = 1$ and $|B_j(T_i)| \leq 1$ since $L(t,x) > 0$. A numeraire $\theta(t)$ is defined by

$$\theta(t) = \frac{B_{m(t)}(t)}{B_1(0)}\prod_{j=1}^{m(t)-1}(1+\delta L(T_j,0)), \quad i \leq n-1, \tag{11}$$

where $m(t)$ is an integer satisfying

$$T_{m(t)-1} < t \leq T_{m(t)}.$$

It is known[5] that $B_i(t)/\theta(t)$ is a Q martingale for all i. Then B is arbitrage-free. In this paper, we call B *the LIBOR market model* after [6]. We denote the set of all LIBOR market models by $\mathcal{L}_Q$. Obviously $\mathcal{L}_Q$ is not included in $\mathcal{H}_Q$, Then we have the following inclusion relation.

$$\mathcal{L}_Q \subset \mathcal{N}_Q \setminus \mathcal{H}_Q$$

Note that in this paper, the BGM model is included in $\mathcal{H}_Q$. Moreover if γ is regular, it holds that

$$B_j(T_i) = P(T_i,T_j), \qquad (i,j) \in J, \tag{12}$$
$$\theta(T_i) = \beta(T_i), \qquad 0 \leq i \leq n. \tag{13}$$

[4] Let $r(t) = r(t,\cdot)$, $\nu(t) = \nu(t,\cdot)$ and $\nu_x(t) = \nu_x(t,\cdot)$. (5) is expressed by

$$dr(t) = \left\{\frac{\partial r(t)}{\partial x} + \nu(t)\cdot\nu_x(t)\right\}dt + \nu_x(t)\cdot dW_t.$$

$\partial/\partial x$ is an infinitesimal generator of a semigroup $S(s)$ such that $S(s)r(t) = r(t, s+\cdot)$. $r(t)$ is said to be a mild solution to the above equation if it holds that

$$r(t) = S(t)r(0) + \int_0^t S(t-s)(\nu_x(s)\cdot\nu(s))\,ds + \int_0^t S(t-s)\nu_x(s)\cdot dW_s.$$

This is equivalent to (6). For details see Da Prato and Zabczyk [3].

[5] For details see [6] or [8].

Remark. We briefly recall the space introduced in the previous work [10], where the space G_{bgm} is defined as the set of volatility functions $\gamma(t,x)$. Obviously G_{bgm} includes only the BGM model, and is not extendable to involve other term structure models, such as the LIBOR market model.

Moreover, the metric on G_{bgm} was defined by a sum of the difference of $\gamma(t,x)$, $L(t,x)$, and prices of some options. This metric is not only complicated but also ambiguous because the metric depends on the choice of options.

On the other hand, our new space $\mathcal{Y}$ is defined as the equivalent class of bond prices at $(T_i,T_j) \in J$. Therefore $\mathcal{Y}$ involves almost all term structure models, including the short rate model, the whole yield curve model like the HJM model, and the discrete family of bonds like the LIBOR market model. Moreover, the metric on $\mathcal{Y}$ is given by the difference of only the bond prices, so the topology of $\mathcal{Y}$ is weaker than that of G_{bgm} Hence $\mathcal{Y}$ is a broad generalization of G_{bgm}.

4. L^2-theoretical relation among term structure models

Let Q be fixed, and let Δ be a closed domain in $\mathbf{R}^2$ defined by

$$\Delta = \{(t,x) \in \mathbf{R}^2 : t,x \geq 0,\ t+x \leq T_n\}$$

Consider a sequence of deterministic volatility functions $\{\gamma_\alpha(t,x)\}_{\alpha\geq 0}$ such that each $\gamma_\alpha(t,x)$ is bounded and piecewise-continuous. $L_\alpha(t,x)$ denotes the LIBOR process associated with $\gamma_\alpha(t,x)$. Let $B_{\alpha j}(t)$, $j=1,\ldots,n$ and $\theta_\alpha(t)$ be a bond price and the numeraire implied from $\gamma_\alpha(t,x)$ by (10) and (11), respectively. Particularly, if $\gamma_\alpha(t,x)$ is regular, then we obtain $P_\alpha(t,T)$ and $\beta_\alpha(t)$ with respect to α. It follows from (12) and (13) that

$$B_{\alpha j}(T_i) = P_\alpha(T_i,T_j), \qquad (i,j) \in J, \tag{14}$$

$$\theta_\alpha(T_i) = \beta_\alpha(T_i), \qquad 0 \leq i \leq n. \tag{15}$$

The following definition is one of the sufficient conditions for the convergence of $L_\alpha(t,x)$.

Definition 2. We say that γ_α converges to γ_0 on Δ in *condition* $\mathcal{L}$ if for arbitrarily small $\varepsilon > 0$ there exists a positive constant α_0 and measurable subsets d_ε^T, $d_\varepsilon^X \subset I$ such that $\mu(d_\varepsilon^T) < \varepsilon$, $\mu(d_\varepsilon^X) < \varepsilon$, and $|\gamma_\alpha - \gamma_0| < \varepsilon$ on a set

$$\Delta \setminus \{(t,x) : t \in d_\varepsilon^T \text{ or } x \in d_\varepsilon^X\}$$

for all $0 < \alpha < \alpha_0$, where $\mu(\ )$ is the usual Lebesgue measure on $\mathbf{R}$.

Note that condition $\mathcal{L}$ is slightly stronger than uniform convergence in the wider sense. The next example shows that condition $\mathcal{L}$ is not a severe restriction within the BGM framework.

Example 4.1. Let $\gamma(t,x)$ be a piecewise constant function on Δ such that $\gamma(t,x)$ is constant on each cell,

$$\Delta_{ij} = \{(t,x) \in \Delta : \delta(i-1) \leq t < \delta i,\ \delta(j-1) \leq x < \delta j\}.$$

Obviously $\gamma(t,x)$ is not regular. We set for $\epsilon > 0$,

$$\gamma_\epsilon(t,x) = \begin{cases} \gamma(t,x) & x > \epsilon \\ 0 & \epsilon \geq x \geq 0. \end{cases}$$

Consider a 2-dimensional convolution operator ϕ_ϵ which is a smooth function with compact support such that the support of ϕ_ϵ reduces to the central point as $\epsilon \to 0$. Since $\gamma(t,x)$ is integrable on Δ, the convolution $\phi_\epsilon * \gamma_\epsilon$ is regular, and $\{\phi_\epsilon * \gamma_\epsilon(t,x)\}_{\epsilon\geq 0}$ converges to $\gamma(t,x)$ in condition $\mathcal{L}$. □

Under the condition $\mathcal{L}$, we have the following result, in which $E_Q[\ \ ]$ denotes the expectation with respect to Q. Note that the proof does not assume the existence of $r(t,x)$.

Proposition 1. *Let $L(0,x)$ be a positive initial LIBOR. Suppose that $\{\gamma_\alpha(t,x)\}_{\alpha\geq 0}$ is a sequence of uniformly bounded, piecewise continuous, and deterministic volatilities. If γ_α converges to γ_0 on Δ in condition $\mathcal{L}$, then it follows that*

$$\lim_{\alpha\to 0} E_Q[|L_\alpha(T_i,T_k) - L_0(T_i,T_k)|^2] = 0, \quad (i,k) \in K, \tag{16}$$

$$\lim_{\alpha\to 0} E_Q[|B_{\alpha j}(T_i) - B_{0j}(T_i)|^2] = 0, \quad (i,j) \in J, \tag{17}$$

$$\lim_{\alpha\to 0} E_Q[|\theta_\alpha(T_i) - \theta_0(T_i)|^2] = 0, \quad 0 \leq i \leq n. \tag{18}$$

Proof. The proof for (16) is given in [10]. Since $L_\alpha(t,x) > 0$ on Δ, it holds for $(i,j) \in J$ that

$$E_Q[|1+\delta L_\alpha(T_i,T_{j-i})|^{-2}] < 1.$$

And from (10), it holds that $0 < B_j(T_i) \leq 1$. Then we have

$$E_Q[|B_j(T_i)|^3] \leq 1.$$

Applying Lemma 1 in the Appendix iteratively for (10), we have (17).

For an arbitrary integer $m \geq 1$, [10] shows that there exists a positive constant C depending on m such that

$$E_Q[|L_\alpha(T_i,T_k)|^m] < C$$

for $(i,k) \in K$. It follows by the Minkowski inequality that

$$E_Q[|1+\delta L_\alpha(T_i,T_k)|^m] < 1 + C. \tag{19}$$

From (11) we have

$$\theta_\alpha(T_i) = \prod_{l=0}^{i-1}(1+\delta L_\alpha(T_l,0)). \tag{20}$$

From (19), (20) and Lemma 2 it follows that

$$E[|\theta_\alpha(T_i)|^3] < C' \tag{21}$$

for a positive constant C' depending on n.

Since

$$(1+\delta L_\alpha(T_i,0)) - (1+\delta L_0(T_i,0)) = \delta(L_\alpha(T_i,0) - L_0(T_i,0)),$$

(16) implies that $(1+\delta L_\alpha(T_i,0))$ converges to $(1+\delta L_0(T_i,0))$ in L^2-sense for all i. Iteratively applying Lemma 1 in the Appendix for (19), (20) and (21), we obtain (18). This completes the proof. □

Additionally, [10] shows that under the condition $\mathcal{L}$, the price convergence holds for a class of options that includes European cap and swaption.

The next theorem shows that the LIBOR market models are placed at the boundary[6] of $\mathcal{H}_Q$. This topologically explains the relation between the LIBOR market model and the HJM model. Note that $\overline{\mathcal{H}_Q}$ is well defined since $\mathcal{Y}_Q$ is complete.

Theorem 1. *It holds that*

$$\mathcal{L}_Q \subset \mathcal{N}_Q \cap (\overline{\mathcal{H}_Q} \setminus \mathcal{H}_Q). \tag{22}$$

Proof. Let $L(0,x) > 0$ be an initial LIBOR and $\gamma_0(t,x)$ be a bounded, piecewise continuous, and deterministic volatility. Also assume that $\gamma_0(t,x)$ is not regular. Then there exists a LIBOR process $L_0(t,x)$ and a bond price process B_0 associated with $\gamma_0(t,x)$. From the definition of the LIBOR market model, it holds that $B_0 \notin \mathcal{H}_Q$. To prove the theorem, it is sufficient to construct a sequence $\{P_\alpha\}$ in $\mathcal{H}_Q$ that converges to B_0 in $\mathcal{Y}_Q$.

By analogy with Example 4.1, we can find a sequence $\gamma_\alpha(t,x)$, $\alpha > 0$ of uniformly bounded, continuous, and regular functions such that $\gamma_\alpha(t,x)$ converges to $\gamma_0(t,x)$ on Δ in condition $\mathcal{L}$ as $\alpha \to 0$. Let $L_\alpha(t,x)$ and $P_\alpha(t,T)$ be the LIBOR and the bond price associated with $\gamma_\alpha(t,x)$, $\alpha > 0$. Then it holds that $P_\alpha \in \mathcal{H}_Q$ for every $\alpha > 0$. Since γ_α is regular when $\alpha > 0$, it follows from (14) and (15) that

$$B_{\alpha j}(T_i) = P_\alpha(T_i, T_j).$$

From Proposition 1, $\lim_{\alpha\to 0} d_Q(P_\alpha, B_0) = 0$. Hence $B_0 \in \overline{\mathcal{H}_Q}$. This completes the proof. □

Remark. By all rights, Theorem 1 should be described under the original measure. With regard to this issue, recall that the BGM model is one of the HJM models, which is obviously constructed under P like as (2). Naturally, the market price of risk explains the relation between P and Q. On the one hand, the LIBOR market model is originally constructed under the risk-neutral measure Q. This is specified under the original measure P in Yasuoka [11], where $d\mathsf{Q}/d\mathsf{P}$ is clarified. Under these relations, it is expected that $\mathcal{Y}_P$ is equivalent to $\mathcal{Y}_Q$. This is a rather technical matter, so our argument is developed under the fixed measure Q for simplicity.

5. Conclusion

We constructed a metric space $\mathcal{Y}$ of bond price processes that admits natural properties in a financial sense. The metric contributes to see property that the LIBOR market model inherits from the HJM model through this metric. Consequently, we obtain the inclusion relation among term structure models, as shown in Figure 1,

$\mathcal{Y}$ involves many types of term structure models, for example Vasicek [9], Cox et al. [2], and Ho and Lee [5]. Hence from a mathematical viewpoint, it would be possible to topologically classify the relation among them.

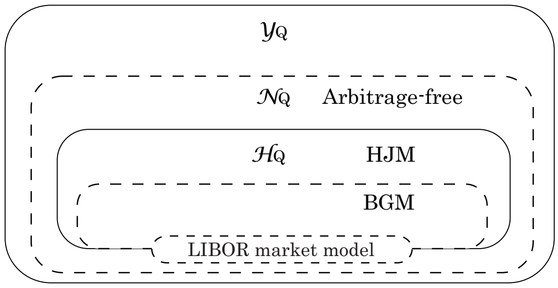

Figure 1: Inclusion relation among term-structure models

Appendix

Lemma 1. *$\{A_\alpha\}$ and $\{B_\alpha\}$ are sequences of stochastic variables that respectively converge to A_0 and B_0 in L^2 sense when α goes to zero. If it holds that*

$$E[|A_\alpha|^2] < C \quad \text{and} \quad E[|A_\alpha B_\alpha|^3] < C$$

for $\alpha > 0$, then $\{A_\alpha B_\alpha\}$ converges to $A_0 B_0$ in L^2 sense.

Proof. The Schwarz inequality implies

$$\begin{aligned}
&E_Q[|A_\alpha B_\alpha - A_0 B_0|] \\
&\quad \le E_Q[|A_\alpha(B_\alpha - B_0)|] + E_Q[|(A_\alpha - A_0)B_0|] \\
&\quad \le E_Q[|A_\alpha|^2]^{\frac{1}{2}} E_Q[(B_\alpha - B_0)^2]^{\frac{1}{2}} \\
&\qquad + E_Q[|B_0|^2]^{\frac{1}{2}} E_Q[(A_\alpha - A_0)^2]^{\frac{1}{2}} \\
&\quad \le C' E_Q[(B_\alpha - B_0)^2]^{\frac{1}{2}} + C' E_Q[(A_\alpha - A_0)^2]^{\frac{1}{2}}
\end{aligned}$$

for some positive constant C'. Then $A_\alpha B_\alpha$ converges to $A_0 B_0$ in probability. From the assumption it follows that

$$E[|A_\alpha B_\alpha|^3] < C'$$

for some positive constant C'. Then from the second inequality in the assumption, $A_\alpha B_\alpha$ converges to $A_0 B_0$ in L^2 sense. □

Lemma 2. *Let A_i, $i = 1, \dots, n$ be stochastic variables. For an arbitrary integer $m > 0$, if there exists a positive constant $C(m)$ depending on m such that*

$$E[|A_i|^m] < C(m)$$

for all i, then it follows that

$$E[|A_1 A_2 \cdots A_j|^3] < C' \tag{23}$$

for arbitrary j, $1 \le j \le n$, where C' is a positive constant depending on n.

[6]The term "boundary" is used expediently. Indeed, $\overline{\mathcal{H}_Q} \setminus \mathcal{H}_Q$ is not exactly a boundary of $\mathcal{H}_Q$, since $\mathcal{H}_Q$ is not necessary an open set in $\mathcal{Y}_Q$.

Proof. Since the cases $i = 1, 2$ are trivial, it is sufficient to prove when $j \geq 3$. Using the Schwarz inequality twice, we have

$$\begin{aligned} E[|A_1A_2A_3|^3]^2 &\leq E[|A_1|^6]E[|A_2A_3|^6] \\ &\leq E[|A_1|^6](E[|A_2|^{12}]E[|A_3|^{12}])^{1/2} \\ &\leq C(6)C(12). \end{aligned}$$

Thus (23) holds for $j = 3$. Similarly, (23) is obtained for $j \geq 4$. □

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