Journal of Math-for-Industry, Vol. 4 (2012A-2), pp. 5–15

On some properties of a discrete hungry Lotka-Volterra system of multiplicative type

Yosuke Hama, Akiko Fukuda, Yusaku Yamamoto, Masashi Iwasaki, Emiko Ishiwata and Yoshimasa Nakamura

Received on October 24, 2011 / Revised on November 17, 2011

Abstract. Two kinds of discrete hungry Lotka-Volterra systems (dhLV) are known as discretizations of the additive type hungry Lotka-Volterra system and the multiplicative one. By associating the dhLV of additive type (dhLV$_{\mathrm{I}}$) and the discrete hungry Toda equation (dhToda) with LR transformations, some of the authors give a Bäcklund transformation between these two systems. In this paper, from the dhLV of multiplicative type (dhLV$_{\mathrm{II}}$), we first derive the qd-type dhLV$_{\mathrm{II}}$. Through finding the positivity of the qd-type dhLV$_{\mathrm{II}}$ and the LR transformation associated with the dhLV$_{\mathrm{II}}$, we present Bäcklund transformations among the dhLV$_{\mathrm{I}}$, the dhLV$_{\mathrm{II}}$ and the dhToda. Moreover, by using one of the Bäcklund transformations, we show asymptotic convergence of the qd-type dhLV$_{\mathrm{II}}$.

Keywords. Bäcklund transformation, LR transformation, asymptotic convergence, discrete hungry Toda equation, discrete hungry Lotka-Volterra system

1. Introduction

The integrable Lotka-Volterra system (LV) is known as one of the ordinary differential equations that describe predator-prey dynamics in mathematical biology. In [1, 2, 3], one of extended LV is presented as

$$\begin{cases} \dfrac{\mathrm{d}u_k(t)}{\mathrm{d}t} = u_k(t)\left(\displaystyle\sum_{p=1}^{M} u_{k+p}(t) - \sum_{p=1}^{M} u_{k-p}(t)\right), \\ \quad k = 1, 2, \ldots, M_m, \quad t \geq 0, \\ u_{1-M}(t) \equiv 0, \ldots, u_0(t) \equiv 0, \\ u_{M_m+1}(t) \equiv 0, \ldots, u_{M_m+M}(t) \equiv 0, \end{cases} \tag{1}$$

and another extended LV is given in [2, 3] as

$$\begin{cases} \dfrac{\mathrm{d}v_k(t)}{\mathrm{d}t} = v_k(t)\left(\displaystyle\prod_{p=1}^{M} v_{k+p}(t) - \prod_{p=1}^{M} v_{k-p}(t)\right), \\ \quad k = 1, 2, \ldots, M_m + M - 1, \quad t \geq 0, \\ v_{1-M}(t) \equiv 0, \ldots, v_0(t) \equiv 0, \\ v_{M_m+M}(t) \equiv 0, \ldots, v_{M_m+M+(M-1)}(t) \equiv 0, \end{cases} \tag{2}$$

where M is a positive integer, $M_k := (M+1)k - M$, and $u_k(t)$ and $v_k(t)$ denote the populations of the kth species at the continuous time t. Eqs. (1) and (2) describe the competition such that the kth species is predator of the $(k+1)$th, the $(k+2)$th, ..., the $(k+M)$th species and is prey of the $(k-1)$th, the $(k-2)$th, ..., the $(k-M)$th species. In the case of $M = 1$, both (1) and (2) become the original LV. As M grows larger, for the kth species, the number of species of both the preys and the predators increase. So, (1) and (2) are called the hungry LV (hLV) of additive type and multiplicative type, respectively. Sometimes, (1) and (2) are referred to as the Bogoyavlensky lattices. The hLV (1) and (2) are also derived from a spatial discretization of the Korteweg-de Vries equation [4].

The discretized version of (1) is presented in [5, 6] as

$$\begin{cases} u_k^{(n+1)} \displaystyle\prod_{p=1}^{M}\left(1 + \delta^{(n+1)} u_{k-p}^{(n+1)}\right) = u_k^{(n)} \prod_{p=1}^{M}\left(1 + \delta^{(n)} u_{k+p}^{(n)}\right), \\ \quad k = 1, 2, \ldots, M_m, \quad n = 0, 1, \ldots, \\ u_{1-M}^{(n)} \equiv 0, \ldots, u_0^{(n)} \equiv 0, \quad u_{M_m+1}^{(n)} \equiv 0, \ldots, u_{M_m+M}^{(n)} \equiv 0, \end{cases} \tag{3}$$

and that of (2) is given in [6] as

$$\begin{cases} v_k^{(n+1)}\left(1 + \delta^{(n+1)} \displaystyle\prod_{p=1}^{M} v_{k-p}^{(n+1)}\right) = v_k^{(n)}\left(1 + \delta^{(n)} \prod_{p=1}^{M} v_{k+p}^{(n)}\right), \\ \quad k = 1, 2, \ldots, M_m + M - 1, \quad n = 0, 1, \ldots, \\ v_{1-M}^{(n)} \equiv 0, \ldots, v_0^{(n)} \equiv 0, \\ v_{M_m+M}^{(n)} \equiv 0, \ldots, v_{M_m+M+(M-1)}^{(n)} \equiv 0, \end{cases} \tag{4}$$

respectively. Both (3) and (4) are called *the discrete hLV* (dhLV). In this paper, in order to distinguish two kinds of the dhLVs, we simply refer to (3) and (4) as *the dhLV*$_{\mathrm{I}}$ associated with the continuous hLV of additive type (1) and *the dhLV*$_{\mathrm{II}}$ associated with the continuous hLV of multiplicative one (2), respectively. In (3) and (4), $\delta^{(n)}$ represents the step size at the discrete time n. The variables $u_k^{(n)}$ and $v_k^{(n)}$ denote the population of the kth species at the discrete time n. The dhLV$_{\rm I}$ (3) is shown in [7] to have an application for computing complex eigenvalues of a certain band matrix.

The discrete Toda equation

$$\begin{cases} q_i^{(n+1)} + e_{i-1}^{(n+1)} = q_i^{(n)} + e_i^{(n)}, \quad i = 1, 2, \ldots, m, \\ q_i^{(n+1)} e_i^{(n+1)} = q_{i+1}^{(n)} e_i^{(n)}, \quad i = 1, 2, \ldots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \quad n = 0, 1, \ldots, \end{cases} \tag{5}$$

is also a famous integrable system. Here, the superscript n is the time variable, as in (3) and (4), and the subscript i denotes the spatial variable. A study on box and ball system in [8] leads to an extended version of the discrete Toda equation (5),

$$\begin{cases} Q_i^{(n+M)} + E_{i-1}^{(n+1)} = Q_i^{(n)} + E_i^{(n)}, \quad i = 1, 2, \ldots, m, \\ Q_i^{(n+M)} E_i^{(n+1)} = Q_{i+1}^{(n)} E_i^{(n)}, \quad i = 1, 2, \ldots, m-1, \\ E_0^{(n)} \equiv 0, \quad E_m^{(n)} \equiv 0, \quad n = 0, 1, \ldots, \end{cases} \tag{6}$$

with positive integer M, which is named *the discrete hungry Toda equation*. In this paper, for the simplicity, we call (6) *the dhToda*. In [9], a new algorithm for computing matrix eigenvalues is designed based on the dhToda (6).

Some of the authors in [10] found a relationship of dependent variables, namely, a Bäcklund transformation, between the dhLV$_{\rm I}$ (3) and the dhToda (6) through associating these integrable systems with a sequence of LR transformations of matrices. Bäcklund transformation is originally derived from the study of differential geometry. Explicit form of the Bäcklund transformation helps us to understand intrinsic features of an integrable system such as the solutions and symmetry and its relationship with another integrable system [11].

Here, for a nonsingular matrix A, the LR transformation [12] is defined as

$$A = LR, \quad \hat{A} = RL. \tag{7}$$

The 1st equation of (7) represents the LR decomposition of A where L is a lower triangular and R is a unit upper triangular matrix. It is to be noted that the LR decomposition where R has unit diagonal entries is uniquely given. The 2nd equation generates $\hat{A}$ as the matrix product RL. Let $\hat{A} = \hat{L}\hat{R}$ be the LR decomposition of $\hat{A}$. From (7), we get $\hat{L}\hat{R} = RL$. This type of equation appears in the matrix representation of some discrete integrable systems, and is called the Lax representation of them. The eigenvalues of $\hat{A}$ coincide with those of A. So, the LR transformation (7) yields a similarity transformation from A to $\hat{A}$, namely, $\hat{A} = RAR^{-1}$. For example, in order to compute the eigenvalues of a symmetric tridiagonal matrix, the quotient difference (qd) algorithm employs a sequence of LR transformations. It is interesting that the recursion formula of the qd algorithm is just equal to the discrete Toda equation (5).

However, there is no observation that the dhLV$_{\rm II}$ (4) is associated with a sequence of LR transformations. In this paper, we first associate the dhLV$_{\rm II}$ (4) with a sequence of LR transformations. Based on this result, we present a Bäcklund transformation between the dhLV$_{\rm II}$ (4) and the dhToda (6). Additionally, a Bäcklund transformation between the dhLV$_{\rm I}$ (3) and the dhLV$_{\rm II}$ (4) is also presented for the case of $\delta^{(n)} \to \infty$.

With the help of the relationship among the dhLV$_{\rm I}$ (3), the dhLV$_{\rm II}$ (4) and the dhToda (6), we next show the asymptotic behavior of the dhLV$_{\rm II}$ (4) as $n \to \infty$, by using the convergence property of the dhToda (6) given in [9]. The dhLV$_{\rm II}$ (4) is also shown to be applicable for matrix eigenvalue computation.

This paper is organaized as follows. In Section 2, we derive a system called *the qd-type dhLV*$_{\rm II}$ from the original dhLV$_{\rm II}$ (4) through variable transformation. We also show the positivity of the qd-type dhLV$_{\rm II}$ under suitable conditions. In Section 3, we give a Lax representation for the dhLV$_{\rm II}$ (4), and then relate it to the LR transformation of a band matrix. We also review the Lax representation for the dhToda (6) and the LR transformation associated with it. In Section 4, by comparing two LR transformations in Section 2, we derive a Bäcklund transformation between the dhLV$_{\rm II}$ (4) and the dhToda (6). By taking account of the Bäcklund transformation between the dhLV$_{\rm I}$ (3) and the dhToda (6) given in [10], we also derive a Bäcklund transformation between the dhLV$_{\rm I}$ (3) and the dhLV$_{\rm II}$ (4) for the case of $\delta^{(n)} \to \infty$. We investigate the asymptotic behaviour of the dhLV$_{\rm II}$ variables through the Bäcklund transformation between the dhLV$_{\rm II}$ (4) and the dhToda (6). The asymptotic behaviour of the dhToda variables is already shown in [9]. In Section 5, we give numerical examples in order to demonstrate some theorems in the previous sections. Finally, in Section 6, conclusion is presented.

2. The qd-type dhLV$_{\rm II}$ and positivity of its variables

In this section, we introduce the qd-type dhLV$_{\rm II}$ which is derived from the dhLV$_{\rm II}$ (4) and show the positivity of its variables. For the simplicity, we employ the notations $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and Φ_5 defined as

$$\begin{aligned} \Phi_1 &:= \{1, 2, \ldots, M_m + M - 1\}, \\ \Phi_2 &:= \{1, 2, \ldots, M_{m-1} + M\}, \\ \Phi_3 &:= \{1, 2, \ldots, m-1\}, \\ \Phi_4 &:= \{1, 2, \ldots, m\}, \\ \Phi_5 &:= \{0, 1, \ldots, M-1\}. \end{aligned}$$

These index sets appear throughout this paper frequently.

2.1. The qd-type dhLV$_{\rm II}$

Let us introduce new variables

$$\omega_k^{(n)} := v_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^{M} v_{k-p}^{(n)} \right), \quad {}^\forall k \in \Phi_1, \tag{8}$$

$$\gamma_k^{(n)} := \delta^{(n)} \prod_{p=0}^{M} v_{k+p}^{(n)}, \quad {}^\forall k \in \Phi_2, \tag{9}$$

from the dhLV$_{\rm II}$ variable $v_k^{(n)}$ and the discrete step size $\delta^{(n)}$. From the boundary condition of $v_k^{(n)}$, we have

$$\omega_k^{(n)} = v_k^{(n)}, \quad {}^\forall k \in \Phi_5 \cup \{M\} \setminus \{0\}, \tag{10}$$
$$\gamma_{-k}^{(n)} = 0, \quad {}^\forall k \in \Phi_5, \tag{11}$$
$$\gamma_{M_m+j}^{(n)} = 0, \quad {}^\forall j \in \Phi_5. \tag{12}$$

Then these variables satisfy the recursion formula

$$\begin{cases} \omega_k^{(n+1)} + \gamma_{k-M}^{(n)} = \omega_k^{(n)} + \gamma_k^{(n)}, & {}^\forall k \in \Phi_1, \\ \omega_k^{(n+1)}\gamma_{k+1}^{(n)} = \omega_{k+M+1}^{(n)}\gamma_k^{(n)}, & {}^\forall k \in \Phi_2 \setminus \{M_{m-1}+M\}. \end{cases} \tag{13}$$

This is easily checked as follows.

$$\begin{aligned} &\omega_k^{(n+1)} + \gamma_{k-M}^{(n)} \\ &= v_k^{(n+1)}\left(1+\delta^{(n+1)}\prod_{p=1}^{M} v_{k-p}^{(n+1)}\right) + \delta^{(n)}\prod_{p=0}^{M} v_{k-M+p}^{(n)} \\ &= v_k^{(n)}\left(1+\delta^{(n)}\prod_{p=1}^{M} v_{k+p}^{(n)}\right) + \delta^{(n)}\prod_{p=0}^{M} v_{k-p}^{(n)} \\ &= v_k^{(n)}\left(1+\delta^{(n)}\prod_{p=1}^{M} v_{k-p}^{(n)}\right) + \delta^{(n)}\prod_{p=0}^{M} v_{k+p}^{(n)} \\ &= \omega_k^{(n)} + \gamma_k^{(n)}, \end{aligned}$$

$$\begin{aligned} &\omega_k^{(n+1)}\gamma_{k+1}^{(n)} \\ &= \left[v_k^{(n+1)}\left(1+\delta^{(n+1)}\prod_{p=1}^{M} v_{k-p}^{(n+1)}\right)\right]\left[\delta^{(n)}\prod_{p=0}^{M} v_{k+p+1}^{(n)}\right] \\ &= \left[v_k^{(n)}\left(1+\delta^{(n)}\prod_{p=1}^{M} v_{k+p}^{(n)}\right)\right]\left[\delta^{(n)}\prod_{p=0}^{M} v_{k+p+1}^{(n)}\right] \\ &= v_{k+M+1}^{(n)}\left(1+\delta^{(n)}\prod_{p=1}^{M} v_{k+p}^{(n)}\right)\left(\delta^{(n)}\prod_{p=0}^{M} v_{k+p}^{(n)}\right) \\ &= v_{k+M+1}^{(n)}\left(1+\delta^{(n)}\prod_{p=1}^{M} v_{k+M+1-p}^{(n)}\right)\left(\delta^{(n)}\prod_{p=0}^{M} v_{k+p}^{(n)}\right) \\ &= \omega_{k+M+1}^{(n)}\gamma_k^{(n)}. \end{aligned}$$

Eq. (13) has the form similar to the recursion formula of the qd algorithm (5). In order to distinguish (13) from the dhLV$_{\rm II}$ (4), we hereinafter call (13) *the qd-type dhLV*$_{\rm II}$. Also, we can rewrite the qd-type dhLV$_{\rm II}$ as

$$\begin{cases} \omega_k^{(n+1)} = \omega_k^{(n)} + \gamma_k^{(n)} - \gamma_{k-M}^{(n)}, \\ \gamma_{k+1}^{(n)} = \dfrac{\omega_{k+M+1}^{(n)}\gamma_k^{(n)}}{\omega_k^{(n+1)}}. \end{cases} \tag{14}$$

If $\omega_k^{(n)}$ for ${}^\forall k \in \Phi_1$ and $\gamma_1^{(n)}$ are given, we can obtain $\omega_k^{(n+1)}$ for ${}^\forall k \in \Phi_1$ and $\gamma_k^{(n)}$ for ${}^\forall k \in \Phi_2 \setminus \{1\}$ by using (14). Let us assume that $\omega_k^{(n)} > 0$ for ${}^\forall k \in \Phi_1$ and $\gamma_1^{(n)} > 0$. Then we can relate $\gamma_1^{(n)}$ to $\delta^{(n)}$ as follows. From (9) and (10), we derive

$$\begin{aligned} \gamma_1^{(n)} &= \delta^{(n)}\prod_{p=1}^{M+1} v_p^{(n)} \\ &= \delta^{(n)} v_{M+1}^{(n)}\prod_{p=1}^{M}\omega_p^{(n)} \\ &= \frac{\delta^{(n)}\omega_{M+1}^{(n)}\prod_{p=1}^{M}\omega_p^{(n)}}{1+\delta^{(n)}\prod_{p=1}^{M}\omega_p^{(n)}} \\ &= \frac{\prod_{p=1}^{M+1}\omega_p^{(n)}}{\dfrac{1}{\delta^{(n)}}+\prod_{p=1}^{M}\omega_p^{(n)}}. \end{aligned} \tag{15}$$

From (15), it holds that

$$\delta^{(n)} = \frac{\gamma_1^{(n)}}{(\omega_{M+1}^{(n)} - \gamma_1^{(n)})\prod_{p=1}^{M}\omega_p^{(n)}}.$$

Hence, the condition $\delta^{(n)} > 0$ is equivalent to

$$0 < \gamma_1^{(n)} < \omega_{M+1}^{(n)}.$$

2.2. Positivity of the qd-type dhLV$_{\rm II}$ variables

We give a theorem concerning the positivity of the qd-type dhLV$_{\rm II}$ variables $\omega_k^{(n)}$ and $\gamma_k^{(n)}$.

Theorem 1. *Let us assume that $\omega_k^{(n)} > 0$, ${}^\forall k \in \Phi_1$ and $0 < \gamma_1^{(n)} < \omega_{M+1}^{(n)}$, then it holds that*

$$\omega_k^{(n+1)} > 0, \quad {}^\forall k \in \Phi_1,$$
$$\gamma_k^{(n)} > 0, \quad {}^\forall k \in \Phi_2.$$

Proof. In the discussion for the positivity of the qd-type dhLV$_{\rm II}$ variables, it is useful to introduce an auxiliary variable $d_k^{(n)}$ defined by

$$d_k^{(n)} = \omega_k^{(n)} - \gamma_{k-M}^{(n)}, \quad {}^\forall k \in \Phi_1. \tag{16}$$

From (11) and (16), it follows that

$$\begin{cases} d_k^{(n)} = \omega_k^{(n)}, & {}^\forall k \in \Phi_5 \cup \{M\} \setminus \{0\}, \\ d_{M+1}^{(n)} = \omega_{M+1}^{(n)} - \gamma_1^{(n)}. \end{cases} \tag{17}$$

By combining (17) with the assumption, we have

$$d_k^{(n)} > 0, \quad {}^\forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}.$$

In terms of $d_k^{(n)}$, we may rewrite the 1st equation of (14) as

$$\omega_k^{(n+1)} = \gamma_k^{(n)} + d_k^{(n)}, \quad {}^\forall k \in \Phi_1. \tag{18}$$

From (14) and (16), we also get the recursion formula for $d_k^{(n)}$ and $d_{k+M+1}^{(n)}$ as follows.

$$\begin{aligned} d_{k+M+1}^{(n)} &= \omega_{k+M+1}^{(n)} - \gamma_{k+1}^{(n)} \\ &= \frac{\omega_{k+M+1}^{(n)}}{\omega_k^{(n+1)}}\left(\omega_k^{(n+1)} - \gamma_k^{(n)}\right) \\ &= \frac{\omega_{k+M+1}^{(n)}}{\omega_k^{(n+1)}} d_k^{(n)}. \end{aligned} \tag{19}$$

From (14), (18) and (19), we get a differential form without subtraction as follows.

$$\begin{cases} \omega_k^{(n+1)} = \gamma_k^{(n)} + d_k^{(n)}, \\ \gamma_{k+1}^{(n)} = \dfrac{\omega_{k+M+1}^{(n)}\gamma_k^{(n)}}{\omega_k^{(n+1)}}, \\ d_{k+M+1}^{(n)} = \dfrac{\omega_{k+M+1}^{(n)}}{\omega_k^{(n+1)}} d_k^{(n)}. \end{cases} \tag{20}$$

By using (20) repeatedly, we can compute $\omega_k^{(n+1)}, \gamma_{k+1}^{(n)}$ and $d_{k+M+1}^{(n)}$ for $k = 1, 2, \ldots$. Since the initial values $\omega_k^{(n)}$ for ${}^\forall k \in \Phi_1$, $\gamma_1^{(n)}$ and $d_k^{(n)}$ for ${}^\forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$ are positive by assumption and there are no subtractions, we can conclude that all the computed variables are positive. □

3. LR transformations associated with the dhLV$_{\rm II}$

In this section, we give a Lax representation of the dhLV$_{\rm II}$ (4), and then present a sequence of LR transformations associated with the dhLV$_{\rm II}$ (4). In addition, we briefly review [10] concerning the LR transformation associated with the dhToda (6).

A Lax representation of the qd-type dhLV$_{\rm II}$ (13) is given by

$$\hat{L}^{(n+1)}\hat{R}^{(n)} = \hat{R}^{(n)}\hat{L}^{(n)}, \tag{21}$$

where

$$\hat{L}^{(n)} = \begin{pmatrix} 0 & \omega_1^{(n)} & & & & & & \\ \vdots & 0 & \omega_2^{(n)} & & & & & \\ 0 & \vdots & \ddots & \ddots & & & & \\ 1 & 0 & & \ddots & \ddots & & & \\ & 1 & \ddots & & \ddots & \ddots & & \\ & & \ddots & \ddots & & \ddots & \omega_{M_m+M-1}^{(n)} \\ & & & 1 & \underbrace{0 \ \cdots \ 0}_{M} & & \end{pmatrix},$$

$$\hat{R}^{(n)} = \begin{pmatrix} 1 & \overbrace{0 \ \cdots \ 0}^{M} & \gamma_1^{(n)} & & & \\ & \ddots & 0 \ \cdots \ 0 & \gamma_2^{(n)} & & \\ & & \ddots \ \ddots & \ddots & \ddots & \\ & & & \ddots \ \ddots & \ddots & \gamma_{M_m-1}^{(n)} \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots \ \ddots & \vdots \\ & & & & \ddots & 0 \\ & & & & & 1 \end{pmatrix}.$$

By focusing on the entries in the both sides of (21), we can get the qd-type dhLV$_{\rm II}$ (13). This means that (21) is a Lax representation of the qd-type dhLV$_{\rm II}$ (13). Of course, the Lax representation (21) with (8) and (9) is just equal to that of the dhLV$_{\rm II}$ (4) in [6].

It is remarkable here that the Lax representation is not always uniquely given. In the following theorem, we present a new Lax representation for the qd-type dhLV$_{\rm II}$ (13).

Theorem 2. *As $\delta^{(n)} \to \infty$, a Lax representation of the qd-type dhLV$_{\rm II}$ (13) becomes*

$$\mathcal{L}_{j+1}^{(n+1)}\mathcal{R}_{j+1}^{(n)} = \mathcal{R}_j^{(n)}\mathcal{L}_{j+1}^{(n)}, \quad j \in \Phi_5, \tag{22}$$

where

$$\mathcal{L}_j^{(n)} = \begin{pmatrix} \omega_{M_1+j-1}^{(n)} & & & \\ 1 & \omega_{M_2+j-1}^{(n)} & & \\ & \ddots & \ddots & \\ & & 1 & \omega_{M_m+j-1}^{(n)} \end{pmatrix}, \tag{23}$$

$$\mathcal{R}_j^{(n)} = \begin{pmatrix} 1 & \gamma_{M_1+j}^{(n)} & & \\ & 1 & \ddots & \\ & & \ddots & \gamma_{M_{m-1}+j}^{(n)} \\ & & & 1 \end{pmatrix}. \tag{24}$$

Proof. In (20), let us assume that $\omega_k^{(n)} > 0$ for ${}^\forall k \in \Phi_1$, $0 < c \leq \gamma_1^{(n)} \leq \omega_{M+1}^{(n)}$, where c is some positive constant. We first show that the qd-type dhLV$_{\rm II}$ variables satisfy the following inequality.

$$\begin{cases} \underline{d}_k \leq d_k^{(n)} \leq \overline{d}_k, \quad {}^\forall k \in \Phi_1 \setminus \{i(M+1)\}, \quad {}^\forall i \in \Phi_3, \\ d_k^{(n)} \leq \overline{d}_k, \quad {}^\forall k \in \{i(M+1)\}, \quad {}^\forall i \in \Phi_3, \\ \underline{\gamma}_k \leq \gamma_k^{(n)} \leq \overline{\gamma}_k, \quad {}^\forall k \in \Phi_2, \\ \underline{\omega}_k \leq \omega_k^{(n+1)} \leq \overline{\omega}_k, \quad {}^\forall k \in \Phi_1, \end{cases} \tag{25}$$

where $\underline{d}_k, \overline{d}_k, \underline{\gamma}_k, \overline{\gamma}_k, \underline{\omega}_k$ and $\overline{\omega}_k$ are some positive constants that do not depend on $\gamma_1^{(n)}$.

We first consider the case where ${}^\forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$. From (17), we have

$$\underline{d}_k \leq d_k^{(n)} \leq \overline{d}_k, \quad {}^\forall k \in \Phi_5 \cup \{M\} \setminus \{0\}, \tag{26}$$

with $\underline{d}_k = \overline{d}_k = \omega_k^{(n)}$. Similary, from (17), we get

$$d_{M+1}^{(n)} \le \overline{d}_{M+1}, \tag{27}$$

with $\overline{d}_{M+1} = \omega_{M+1}^{(n)}$. Obviously, the assumption leads to

$$\underline{\gamma}_1 \le \gamma_1^{(n)} \le \overline{\gamma}_1, \tag{28}$$

with $\underline{\gamma}_1 = c$ and $\overline{\gamma}_1 = \omega_{M+1}^{(n)}$. By combining (26)–(28) with (20), we can prove the following inequalities by induction.

$$\begin{cases} \underline{\gamma}_k \le \gamma_k^{(n)} \le \overline{\gamma}_k, & \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0,1\}, \\ \underline{\omega}_k \le \omega_k^{(n+1)} \le \overline{\omega}_k, & \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}. \end{cases} \tag{29}$$

In the case where $\forall k \in \{i+M+1\}, \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$, from the 3rd equation of (20), (26), (27) and the 2nd equation of (29), it holds that

$$\underline{d}_k \le d_k^{(n)} \le \overline{d}_k, \quad \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M\} \setminus \{0\}, \tag{30}$$

$$d_{2(M+1)}^{(n)} \le \overline{d}_{2(M+1)}. \tag{31}$$

Moreover, from (29), it follows that

$$\underline{\gamma}_{M+2} \le \gamma_{M+2}^{(n)} \le \overline{\gamma}_{M+2}. \tag{32}$$

Eqs. (30)–(32) lead to

$$\begin{cases} \underline{\gamma}_k \le \gamma_k^{(n)} \le \overline{\gamma}_k, \\ \quad \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0,1\}, \\ \underline{\omega}_k \le \omega_k^{(n+1)} \le \overline{\omega}_k, \\ \quad \forall k \in \{i+M+1\}, \quad \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}. \end{cases}$$

Similary, for $\forall k \in \{(i-1)(M+1)+j+1\}, \forall i \in \Phi_3 \setminus \{1,2\}, \forall j \in \Phi_5$, it follows that

$$\begin{cases} \underline{d}_k \le d_k^{(n)} \le \overline{d}_k, \\ \underline{\gamma}_k \le \gamma_k^{(n)} \le \overline{\gamma}_k, \\ \underline{\omega}_k \le \omega_k^{(n+1)} \le \overline{\omega}_k, \end{cases} \tag{33}$$

and for $\forall k \in \{i(M+1)\}, \forall i \in \Phi_3 \setminus \{1,2\}$, we have

$$d_k^{(n)} \le \overline{d}_k.$$

We next consider the case where $\forall k \in \{(m-1)(M+1)+i+1\}, \forall i \in \Phi_5$. By combining the 3rd equation of (20), (33) with (12), we have

$$\underline{d}_{k+i} \le d_{k+i}^{(n)} \le \overline{d}_{k+i}, \quad \forall i \in \Phi_5. \tag{34}$$

From (34) and the 1st equation of (20), we have

$$\underline{\omega}_{k+i} \le \omega_{k+i}^{(n+1)} \le \overline{\omega}_{k+i}, \quad \forall i \in \Phi_5.$$

To sum up, we obtain (25).

By using (25), we discuss the behavior of variables in (13) as $\delta^{(n)} \to \infty$. We first consider the case of $k = M_i + M$, $\forall i \in \Phi_3$. By using (19) repeatedly, we derive

$$d_{M_i+M}^{(n)} = \frac{\prod_{p=2}^{i} \omega_{M_p+M}^{(n)}}{\prod_{p=1}^{i-1} \omega_{M_p+M}^{(n+1)}} d_{M+1}^{(n)}, \quad \forall i \in \Phi_3 \setminus \{1\}. \tag{35}$$

Eqs. (15) and (17) lead to

$$\begin{aligned} d_{M+1}^{(n)} &= \omega_{M+1}^{(n)} - \gamma_1^{(n)} \\ &= \omega_{M+1}^{(n)} - \frac{\prod_{p=1}^{M+1} \omega_p^{(n)}}{\frac{1}{\delta^{(n)}} + \prod_{p=1}^{M} \omega_p^{(n)}} \\ &= \frac{\omega_{M+1}^{(n)}}{\delta^{(n)} \prod_{p=1}^{M} \omega_p^{(n)} + 1}. \end{aligned} \tag{36}$$

As $\delta^{(n)} \to \infty$, from (25), it follows that $d_{M+1}^{(n)} \to 0$ in (36). From (35), we derive $d_{M_i+M}^{(n)} \to 0$ for $\forall i \in \Phi_3 \setminus \{1\}$. By combining them with (16), we get

$$\lim_{\delta^{(n)} \to \infty} \left(\omega_{M_i+M}^{(n)} - \gamma_{M_i}^{(n)} \right) = 0, \quad \forall i \in \Phi_3. \tag{37}$$

Moreover, from (18) and $d_{M_i+M}^{(n)} \to 0$ for $\forall i \in \Phi_3$, we have

$$\lim_{\delta^{(n)} \to \infty} \left(\omega_{M_i+M}^{(n+1)} - \gamma_{M_i+M}^{(n)} \right) = 0, \quad \forall i \in \Phi_3. \tag{38}$$

Thus, as $\delta^{(n)} \to \infty$, (13) becomes the trivial equalities $\omega_{M_i+M}^{(n+1)} + \gamma_{M_i}^{(n)} = \omega_{M_i+M}^{(n)} + \gamma_{M_i+M}^{(n)}$, $\forall i \in \Phi_3$ and $\omega_{M_i+M}^{(n+1)} \gamma_{M_{i+1}}^{(n)} = \omega_{M_{i+1}+M}^{(n)} \gamma_{M_i+M}^{(n)}$, $\forall i \in \Phi_3 \setminus \{m-1\}$.

Next, we consider the cases except for $k = M_i + M$, $\forall i \in \Phi_3$ in the 1st and 2nd equations of (13). We here focus on the product of $\mathcal{L}_{j+1}^{(n+1)}$ and $\mathcal{R}_{j+1}^{(n)}$. The (i,i) and $(i,i+1)$ entries of $\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)}$ are given as, respectively,

$$(\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)})_{i,i} = \omega_{M_i+j}^{(n+1)} + \gamma_{M_i-M+j}^{(n)}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \tag{39}$$

$$(\mathcal{L}_{j+1}^{(n+1)} \mathcal{R}_{j+1}^{(n)})_{i,i+1} = \omega_{M_i+j}^{(n+1)} \gamma_{M_i+j+1}^{(n)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \tag{40}$$

Similarly, it follows that

$$(\mathcal{R}_j^{(n)} \mathcal{L}_{j+1}^{(n)})_{i,i} = \omega_{M_i+j}^{(n)} + \gamma_{M_i+j}^{(n)}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \tag{41}$$

$$(\mathcal{R}_j^{(n)} \mathcal{L}_{j+1}^{(n)})_{i,i+1} = \omega_{M_i+M+j+1}^{(n)} \gamma_{M_i+j}^{(n)}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5. \tag{42}$$

Eqs. (22), (39) and (41) bring to the 1st equation of (13). Also, (22), (40) and (42) lead to the 2nd one. This indicates that (22) is a Lax representation for the qd-type dhLV$_{\text{II}}$ (13). □

Moreover, we give a lemma concerning the relationship of the Lax matrices $\mathcal{R}_0^{(n+1)}$ and $\mathcal{R}_M^{(n)}$ as $\delta^{(n)} \to \infty$.

Lemma 1. *As $\delta^{(n)} \to \infty$, it holds that*

$$\mathcal{R}_0^{(n+1)} = \mathcal{R}_M^{(n)}. \tag{43}$$

Proof. Obviously, from (37) and (38), $\omega_{M_i+M}^{(n+1)} \to \gamma_{M_i}^{(n+1)}$ and $\omega_{M_i+M}^{(n+1)} \to \gamma_{M_i+M}^{(n)}$ as $\delta^{(n)} \to \infty$. So, it holds that $\gamma_{M_i}^{(n+1)} \to \gamma_{M_i+M}^{(n)}$ as $\delta^{(n)} \to \infty$. This leads to (43). □

Let us introduce the matrix, given by the product of the Lax matrices $\mathcal{L}_1^{(n)}, \mathcal{L}_2^{(n)}, \ldots, \mathcal{L}_M^{(n)}$ in (23) and $\mathcal{R}_0^{(n)}$ in (24),

$$\mathcal{A}^{(n)} = \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)} \mathcal{R}_0^{(n)}. \tag{44}$$

Let us consider $\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1}$ as a similarity transformation of $\mathcal{A}^{(n)}$ by $\mathcal{R}_0^{(n)}$. Then, with the help of Theorem 2, we derive

$$\begin{aligned}
\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} &= \mathcal{R}_0^{(n)} \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)} \\
&= \mathcal{L}_1^{(n+1)} \mathcal{R}_1^{(n)} \mathcal{L}_2^{(n)} \mathcal{L}_3^{(n)} \cdots \mathcal{L}_M^{(n)} \\
&= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \mathcal{R}_2^{(n)} \mathcal{L}_3^{(n)} \cdots \mathcal{L}_M^{(n)} \\
&\ \ \vdots \\
&= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \cdots \mathcal{L}_{M-1}^{(n+1)} \mathcal{R}_{M-1}^{(n)} \mathcal{L}_M^{(n)} \\
&= \mathcal{L}_1^{(n+1)} \mathcal{L}_2^{(n+1)} \cdots \mathcal{L}_{M-1}^{(n+1)} \mathcal{L}_M^{(n+1)} \mathcal{R}_M^{(n)}.
\end{aligned} \tag{45}$$

By combining it with Lemma 1, we see that

$$\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} = \mathcal{A}^{(n+1)}. \tag{46}$$

This means that the eigenvalues of $\mathcal{A}^{(n)}$ are invariant under the time evolution from n to $n+1$. Eqs. (45) and (46) also lead to

$$\begin{cases} \mathcal{A}^{(n)} = (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)}) \mathcal{R}_0^{(n)}, \\ \mathcal{A}^{(n+1)} = \mathcal{R}_0^{(n)} (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)}). \end{cases} \tag{47}$$

Hence, we know that $\mathcal{A}^{(n+1)}$ is given through the LR transformation of $\mathcal{A}^{(n)}$. According to (45), the LR transformation in (47) coincides with M times LR transformations in (22). Let us recall here that (22) is a Lax representation for the dhLV$_{\rm II}$ (4) with $\delta^{(n)} \to \infty$. We therefore have the following theorem.

Theorem 3. *The dhLV$_{\rm II}$ (4) with $\delta^{(n)} \to \infty$ generates the LR transformation from $\mathcal{A}^{(n)}$ to $\mathcal{A}^{(n+1)}$ as in (47).*

According to [8], the dhToda (6) satisfies the Lax representation,

$$L^{(n+M)} R^{(n+1)} = R^{(n)} L^{(n)}, \tag{48}$$

$$L^{(n)} = \begin{pmatrix} Q_1^{(n)} & & & \\ 1 & Q_2^{(n)} & & \\ & \ddots & \ddots & \\ & & 1 & Q_m^{(n)} \end{pmatrix}, \tag{49}$$

$$R^{(n)} = \begin{pmatrix} 1 & E_1^{(n)} & & \\ & 1 & \ddots & \\ & & \ddots & E_{m-1}^{(n)} \\ & & & 1 \end{pmatrix}, \tag{50}$$

where $Q_i^{(n)} > 0$, $^\forall i \in \Phi_4$ and $E_i^{(n)} > 0$, $^\forall i \in \Phi_3$. The Lax representation (48) may look different from that in [8]. Actually, we can easily get the same Lax representation as in [8] through matrix transposition on both sides of (48).

Let $A^{(n)}$ be the product of the Lax matrices $L^{(n)}, L^{(n+1)}, \ldots, L^{(n+M-1)}$ in (49) and $R^{(n)}$ in (50), namely,

$$A^{(n)} = L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)} R^{(n)}. \tag{51}$$

Then, from (48), it follows that

$$\begin{aligned}
&R^{(n)} A^{(n)} (R^{(n)})^{-1} \\
&\quad = R^{(n)} L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)} \\
&\quad = L^{(n+M)} R^{(n+1)} L^{(n+1)} L^{(n+2)} \cdots L^{(n+M-1)} \\
&\quad = L^{(n+M)} L^{(n+M+1)} R^{(n+2)} L^{(n+2)} \cdots L^{(n+M-1)} \\
&\quad\ \ \vdots \\
&\quad = L^{(n+M)} L^{(n+M+1)} \cdots L^{(n+2M-1)} R^{(n+M)} \\
&\quad = A^{(n+M)}.
\end{aligned} \tag{52}$$

Obviously, from (52), the dhToda (6) gives the similarity transformation from $A^{(n)}$ to $A^{(n+M)}$. Eq. (52) is also rewritten as

$$\begin{cases} A^{(n)} = (L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)}) R^{(n)}, \\ A^{(n+M)} = R^{(n)} (L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)}). \end{cases} \tag{53}$$

Thus, the dhToda (6) has a relationship with the LR transformation as follows.

Theorem 4 ([10])**.** *The dhToda (6) generates the LR transformation from $A^{(n)}$ to $A^{(n+M)}$ as in (53).*

4. Bäcklund transformations among the discrete hungry systems

In this seciton, by considering the relationship between the two LR transformations associated with the dhLV$_{\rm II}$ (4) and the dhToda (6), we give a Bäcklund transfomation between the dhLV$_{\rm II}$ (4) and the dhToda (6). By referring to [10], we establish a Bäcklund transformation between the dhLV$_{\rm I}$ (3) and the dhLV$_{\rm II}$ (4). We also investigate the asymptotic behavior of the qd-type dhLV$_{\rm II}$ (13) with the help of the obtained Bäcklund transformation.

4.1. The Bäcklund transformation between the dhLV$_{\rm II}$ and the dhToda

We first show the relationship of the matrices in two LR transformations associated with the dhLV$_{\rm II}$ (4) and the dhToda (6).

Lemma 2. *For some fixed n, let $\mathcal{R}_0^{(n)} = R^{(n)}$ and $\mathcal{L}_{j+1}^{(n)} = L^{(n+j)}$, $^\forall j \in \Phi_5$. Then, it holds that*

$$\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}, \quad \mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}, \quad ^\forall j \in \Phi_5.$$

Proof. The assumption leads to $\mathcal{R}_0^{(n)}\mathcal{L}_1^{(n)} = R^{(n)}L^{(n)}$. Let us recall that $\mathcal{R}_0^{(n)}\mathcal{L}_1^{(n)} = \mathcal{L}_1^{(n+1)}\mathcal{R}_1^{(n)}$ in (22) and $R^{(n)}L^{(n)} = L^{(n+M)}R^{(n+1)}$ in (48). So, it follows that

$$\mathcal{L}_1^{(n+1)}\mathcal{R}_1^{(n)} = L^{(n+M)}R^{(n+1)}.$$

Recall that the upper bidiagonal matrices $\mathcal{R}_1^{(n)}$ and $R^{(n+1)}$ have 1 in every diagonal entry. Hence, by taking account of the uniqueness of LR decomposition, we get

$$\mathcal{L}_1^{(n+1)} = L^{(n+M)}, \quad \mathcal{R}_1^{(n)} = R^{(n+1)}.$$

Similarly, it is easily proved by induction for $j = 1, 2, \ldots, M-1$ that $\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}$ and $\mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}$. □

From (44) and (51), it is obvious that $\mathcal{A}^{(n)} = A^{(n)}$ if $\mathcal{L}_{j+1}^{(n)} = L^{(n+j)}$, $^\forall j \in \Phi_5$ and $\mathcal{R}_0^{(n)} = R^{(n)}$. So, by using Lemma 2, we see that $\mathcal{A}^{(n+1)} = A^{(n+M)}$ since $\mathcal{R}_{j+1}^{(n)} = R^{(n+j+1)}$, $^\forall j \in \Phi_5$ and $\mathcal{L}_{j+1}^{(n+1)} = L^{(n+M+j)}$, $^\forall j \in \Phi_5$. In other words, the evolution from n to $n+1$ of the dhLV$_{\rm II}$ (4) can generate the LR transformation given by the evolution from n to $n+M$ of the dhToda (6).

Let us replace n with $\ell M + j$ in the superscripts of the dhLV$_{\rm II}$ and the dhToda variables. Hereinafter, we consider the evolution from ℓ to $\ell + 1$ by the dhLV$_{\rm II}$ (4) and the dhToda (6). Let us assume that, for some fixed ℓ,

$$\begin{cases} \mathcal{L}_{j+1}^{(\ell)} = L^{(\ell M+j)}, \quad ^\forall j \in \Phi_5, \\ \mathcal{R}_0^{(\ell)} = R^{(\ell M)}. \end{cases} \tag{54}$$

Then, from Lemma 2, it follows that,

$$\mathcal{L}_{j+1}^{(\ell+1)} = L^{((\ell+1)M+j)}, \quad \mathcal{R}_{j+1}^{(\ell)} = R^{(\ell M+j+1)}, \quad ^\forall j \in \Phi_5. \tag{55}$$

From the 2nd equation of (55) and Lemma 1, it holds that $\mathcal{R}_0^{(\ell+1)} = R^{((\ell+1)M)}$. By focusing on the entries of matrices in (54), we derive

$$\begin{cases} E_i^{(\ell M)} = \gamma_{M_i}^{(\ell)}, \quad ^\forall i \in \Phi_3, \\ Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}, \quad ^\forall i \in \Phi_4, \quad ^\forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \ldots$. Moreover, from (55), we obtain

$$E_i^{(\ell M+j+1)} = \gamma_{M_i+j+1}^{(\ell)}, \quad ^\forall i \in \Phi_3, \quad ^\forall j \in \Phi_5,$$

for $\ell = 0, 1, \ldots$. To sum up, we derive a theorem on the relationship of the variables, namely, the Bäcklund transformation, between the qd-type dhLV$_{\rm II}$ (13) and the dhToda (6).

Theorem 5. *A Bäcklund transformation between the qd-type dhLV$_{\rm II}$ (13) with $\delta^{(\ell)} \to \infty$ and the dhToda (6) is given by*

$$\begin{cases} E_i^{(\ell M+j)} = \gamma_{M_i+j}^{(\ell)}, \quad ^\forall i \in \Phi_3, \quad ^\forall j \in \Phi_5, \\ Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}, \quad ^\forall i \in \Phi_4, \quad ^\forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \ldots$.

It is observed that (8) and (9) are the Bäcklund transformation between the qd-type dhLV$_{\rm II}$ (13) and the original dhLV$_{\rm II}$ (4). So, by combining it with Theorem 5, we have a main theorem in this paper.

Theorem 6. *A Bäcklund transformation between the dhLV$_{\rm II}$ (4) with $\delta^{(\ell)} \to \infty$ and the dhToda (6) is given by*

$$\begin{cases} E_i^{(\ell M+j)} = \delta^{(\ell)} \prod_{p=0}^{M} v_{M_i+j+p}^{(\ell)}, \quad ^\forall i \in \Phi_3, \quad ^\forall j \in \Phi_5, \\ Q_i^{(\ell M+j)} = v_{M_i+j}^{(\ell)} \left(1 + \delta^{(\ell)} \prod_{p=1}^{M} v_{M_i+j-p}^{(\ell)}\right), \\ \quad ^\forall i \in \Phi_4, \quad ^\forall j \in \Phi_5, \end{cases}$$

for $\ell = 0, 1, \ldots$.

4.2. The Bäcklund transformation between the dhLV$_{\rm I}$ and the dhLV$_{\rm II}$

Let us introduce the new variables

$$\begin{cases} U_k^{(n)} = u_k^{(n)} \prod_{p=1}^{M} \left(1 + \delta^{(n)} u_{k-p}^{(n)}\right), \quad ^\forall k \in \Phi_2 \cup \{M_m\}, \\ V_k^{(n)} = \dfrac{1}{\delta^{(n)}} \prod_{p=0}^{M} \left(1 + \delta^{(n)} u_{k-p}^{(n)}\right), \quad ^\forall k \in \Phi_1 \cup \{M_m + M\}, \end{cases} \tag{56}$$

in terms of the dhLV$_{\rm I}$ variable $u_k^{(n)}$. Then the dhLV$_{\rm I}$ (3) can be rewritten as

$$\begin{cases} U_k^{(n+1)} + V_{M+k+1}^{(n)} = U_{M+k+1}^{(n)} + V_{M+k}^{(n)}, \quad ^\forall k \in \Phi_2, \\ U_k^{(n+1)} V_k^{(n)} = U_k^{(n)} V_{M+k}^{(n)}, \quad ^\forall k \in \Phi_2 \cup \{M_m\}, \\ U_{M_m+j+1}^{(n)} := 0, \quad V_{M_m+M+j+1}^{(n)} := \dfrac{1}{\delta^{(n)}}, \quad ^\forall j \in \Phi_5. \end{cases} \tag{57}$$

Eq. (57) is named the qd-type dhLV$_{\rm I}$ in [13]. Eq. (56) is a Bäcklund transformation between the original dhLV$_{\rm I}$ (3) and the qd-type dhLV$_{\rm I}$ (57). Some of the authors, in [10], give a Bäcklund transformation between the qd-type dhLV$_{\rm I}$ (57) with $\delta^{(n)} \to \infty$ and the dhToda (6) as follows.

$$\begin{cases} U_{M_i}^{(\ell)} = \prod_{p=0}^{M-1} Q_i^{(\ell M+p)}, \quad {}^\forall i \in \Phi_4, \\ U_{M_i+j+1}^{(\ell)} = \left(\prod_{p=0}^{M-2} Q_i^{(\ell M+p+j+1)}\right) E_i^{(\ell M+j)}, \\ \quad {}^\forall i \in \Phi_3, \quad {}^\forall j \in \Phi_5. \end{cases} \tag{58}$$

By combining the 1st equation of (58) with Theorem 5, for $\ell = 0, 1, \dots,$ we get

$$U_{M_i}^{(\ell)} = \prod_{p=0}^{M-1} \omega_{M_i+p}^{(\ell)}, \quad {}^\forall i \in \Phi_4. \tag{59}$$

Similarly, from Theorem 5 and the 2nd of (58), we derive

$$\begin{aligned} &U_{M_i+j+1}^{(\ell)} \\ &= Q_i^{(\ell M+j+1)} Q_i^{(\ell M+j+2)} \cdots Q_i^{(\ell M+M-1)} \\ &\quad \times Q_i^{((\ell+1)M)} Q_i^{((\ell+1)M+1)} Q_i^{((\ell+1)M+2)} \cdots Q_i^{((\ell+1)M+j-2)} \\ &\quad \times \left(Q_i^{((\ell+1)M+j-1)} E_i^{(\ell M+j)}\right) \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \cdots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \omega_{M_i}^{(\ell+1)} \omega_{M_i+1}^{(\ell+1)} \omega_{M_i+2}^{(\ell+1)} \cdots \omega_{M_i+j-2}^{(\ell+1)} \\ &\quad \times \left(\omega_{M_i+j-1}^{(\ell+1)} \gamma_{M_i+j}^{(\ell)}\right). \end{aligned}$$

By taking account of the 2nd equation of (13), we successively rewrite $U_{M_i+j+1}^{(\ell)}$ as

$$\begin{aligned} U_{M_i+j+1}^{(\ell)} &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \cdots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \omega_{M_i}^{(\ell+1)} \omega_{M_i+1}^{(\ell+1)} \omega_{M_i+2}^{(\ell+1)} \cdots \omega_{M_i+j-2}^{(\ell+1)} \\ &\quad \times \left(\gamma_{M_i+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)}\right) \\ &\quad \vdots \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \cdots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \left(\omega_{M_i}^{(\ell+1)} \gamma_{M_i+1}^{(\ell)}\right) \omega_{M_i+M+2}^{(\ell)} \cdots \omega_{M_i+M+j-2}^{(\ell)} \\ &\quad \times \omega_{M_i+M+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)} \\ &= \omega_{M_i+j+1}^{(\ell)} \omega_{M_i+j+2}^{(\ell)} \cdots \omega_{M_i+M-1}^{(\ell)} \\ &\quad \times \left(\gamma_{M_i}^{(\ell)} \omega_{M_i+M+1}^{(\ell)}\right) \omega_{M_i+M+2}^{(\ell)} \cdots \omega_{M_i+M+j-2}^{(\ell)} \\ &\quad \times \omega_{M_i+M+j-1}^{(\ell)} \omega_{M_i+M+j}^{(\ell)}. \end{aligned}$$

Note here that $\gamma_{M_i}^{(\ell)} \to \omega_{M_i+M}^{(\ell)}$ as $\delta^{(n)} \to \infty$. So, it follows that

$$U_{M_i+j+1}^{(\ell)} = \prod_{p=0}^{M-1} \omega_{M_i+p+j+1}^{(\ell)}, \quad {}^\forall i \in \Phi_3, \quad {}^\forall j \in \Phi_5. \tag{60}$$

From (59) and (60), we have

$$U_k^{(n)} = \prod_{p=0}^{M-1} \omega_{k+p}^{(n)}, \quad {}^\forall k \in \Phi_2 \cup \{M_m\}. \tag{61}$$

By combinig (61) with (8) and (56), it follows that

$$\begin{aligned} &u_k^{(n)} \prod_{p=1}^{M} (1 + \delta^{(n)} u_{k-p}^{(n)}) \\ &\quad = \prod_{p=0}^{M-1} \left(v_{k+p}^{(n)} \left(1 + \delta^{(n)} \prod_{r=1}^{M} v_{k+p-r}^{(n)}\right)\right) \\ &\quad = \left(\prod_{p=0}^{M-1} v_{k+p}^{(n)}\right) \prod_{p=0}^{M-1} \left(1 + \delta^{(n)} \prod_{r=1}^{M} v_{k+p-r}^{(n)}\right) \\ &\quad = \left(\prod_{p=0}^{M-1} v_{k+p}^{(n)}\right) \prod_{p=1}^{M} \left(1 + \delta^{(n)} \prod_{r=0}^{M-1} v_{k+r-p}^{(n)}\right). \end{aligned} \tag{62}$$

From the boundary condition of $u_k^{(n)}$ and $v_k^{(n)}$, the case where $k = 1$ in (62) leads to

$$u_1^{(n)} = \prod_{p=0}^{M-1} v_{1+p}^{(n)}.$$

Similarly, by considering the cases where $k = 2, 3, \dots, M_m$, we have $u_k^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)}$.

The above discussion leads to the following theorem.

Theorem 7. *As $\delta^{(n)} \to \infty$, a Bäcklund transformation between the dhLV$_{\rm I}$ (3) and the dhLV$_{\rm II}$ (4) is given by*

$$u_k^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)}, \quad {}^\forall k \in \Phi_2 \cup \{M_m\},$$

for $n = 0, 1, \dots$.

4.3. The asymptotic behavior of the qd-type dhLV$_{\rm II}$ variables

We next clarify the asymptotic behavior of the qd-type dhLV$_{\rm II}$ variables by combining Theorem 5 with the asymptotic behavior of the dhToda variables given in [9]. Let us again replace n with $\ell M + j$, ${}^\forall j \in \Phi_5$ in the superscript of the dhToda variable. Of course, the limit of $\ell \to \infty$ is equivalent to that of $n \to \infty$. A minor change of the limit in [9] brings to the following theorem with respect to the convergence of the dhToda variables.

Theorem 8 (cf.[9]). *Let $Q_i^{(0)} > 0, Q_i^{(1)} > 0, \dots, Q_i^{(M-1)} > 0$, ${}^\forall i \in \Phi_4$ and $E_i^{(0)} > 0$, ${}^\forall i \in \Phi_3$. As $\ell \to \infty$, the limits of $Q_i^{(\ell M+j)}$ and $E_i^{(\ell M+j)}$ are given by*

$$\lim_{\ell\to\infty} \prod_{p=0}^{M-1} Q_i^{(\ell M+p)} = C_i, \quad {}^\forall i \in \Phi_4,$$

$$\lim_{\ell\to\infty} E_i^{(\ell M+j)} = 0, \quad {}^\forall i \in \Phi_3, \quad {}^\forall j \in \Phi_5, \tag{63}$$

where C_i is some constant and $C_1 \geq C_2 \geq \cdots \geq C_m > 0$.

In [9], it is also shown that $\{Q_i^{(\ell M+j)}\}_{\ell=0,1,\dots}$ is a Cauchy sequence. This implies that $Q_i^{(\ell M+j)}$ converges to some constant $C_{i,j} > 0$ as $\ell \to \infty$, namely,

$$\lim_{\ell\to\infty} Q_i^{(\ell M+j)} = C_{i,j}, \quad {}^\forall i \in \Phi_4, \quad {}^\forall j \in \Phi_5. \tag{64}$$

Since it is shown in Theorem 5 that $Q_i^{(\ell M+j)} = \omega_{M_i+j}^{(\ell)}$, by using it in (64), we get

$$\lim_{\ell\to\infty} \omega_{M_i+j}^{(\ell)} = C_{i,j}, \quad {}^{\forall}i \in \Phi_4, \quad {}^{\forall}j \in \Phi_5. \tag{65}$$

By taking account that $E_i^{(\ell M+j)} = \gamma_{M_i+j}^{(\ell)}$ shown in Theorem 5, from (63), we derive

$$\lim_{\ell\to\infty} \gamma_{M_i+j}^{(\ell)} = 0, \quad {}^{\forall}i \in \Phi_3, \quad {}^{\forall}j \in \Phi_5. \tag{66}$$

It is remarkable that (65) and (66) show the convergence of $\omega_k^{(n)}$ and $\gamma_k^{(n)}$ except for $k = M_i+M$, ${}^{\forall}i \in \Phi_3$ as $\ell \to \infty$. We next study the convergence of $\omega_{M_i+M}^{(\ell)}$ and $\gamma_{M_i+M}^{(\ell)}$. Eqs. (37) and (66) with $j = 0$ leads to

$$\lim_{\ell\to\infty} \omega_{M_i+M}^{(\ell)} = 0, \quad {}^{\forall}i \in \Phi_3. \tag{67}$$

From (38) and (67), it follows that

$$\lim_{\ell\to\infty} \gamma_{M_i+M}^{(\ell)} = 0, \quad {}^{\forall}i \in \Phi_3.$$

We summarize the asymptotic behavior of the qd-type dhLV$_{\mathrm{II}}$ variables as follows.

Theorem 9. *Let us assume that $\omega_k^{(0)} > 0$, ${}^{\forall}k \in \Phi_1$. Then, the limits of the qd-type dhLV$_{\mathrm{II}}$ variables as $\delta^{(n)} \to \infty$ are*

$$\lim_{n\to\infty} \omega_{M_i+j}^{(n)} = C_{i,j}, \quad {}^{\forall}i \in \Phi_4, \quad {}^{\forall}j \in \Phi_5,$$
$$\lim_{n\to\infty} \omega_{M_i+M}^{(n)} = 0, \quad {}^{\forall}i \in \Phi_3,$$
$$\lim_{n\to\infty} \gamma_k^{(n)} = 0, \quad {}^{\forall}k \in \Phi_2.$$

5. Numerical Examples

In this section, we numerically observe some properties of the qd-type dhLV$_{\mathrm{II}}$ (13) shown in the previous sections. We easily realize numerical properties of the original dhLV$_{\mathrm{II}}$ (4) through those of the qd-type dhLV$_{\mathrm{II}}$.

We first demonstrate the asymptotic behavior of the qd-type dhLV$_{\mathrm{II}}$ (13) shown in Theorem 9 numerically. Let $\omega_1^{(0)} = \omega_2^{(0)} = 5$, $\omega_3^{(0)} = \omega_4^{(0)} = \omega_5^{(0)} = 2$, $\omega_6^{(0)} = \omega_7^{(0)} = \omega_8^{(0)} = 1$ and $M = 2, m = 3$, $\delta^{(n)} = 10^{12}$, respectively, in the qd-type dhLV$_{\mathrm{II}}$ (13). It is emphasized here that the qd-type dhLV$_{\mathrm{II}}$ variables, except for $\omega_k^{(0)}$, depend on the value of $\delta^{(n)}$ through $\gamma_1^{(n)}$, as shown in (15). Figures 1 and 2 show the behavior of $\omega_k^{(n)}$, $k = 1,2,4,5,7,8$ and $\omega_k^{(n)}$, $k = 3,6$, $\gamma_k^{(n)}$, $k = 1,2,3,4,5,6$, for $n = 0,1,\ldots,19$, respectively. We see from Figures 1 and 2 that, as n becomes lager, $\omega_k^{(n)}$, $k = 1,2,4,5,7,8$ and $\omega_k^{(n)}$, $k = 3,6$ approach some positive constants and zero, respectively. This numerical result agrees with Theorem 9.

We next give a numerical example in order to confirm the Bäcklund transformation, shown in Theorem 5, between the dhLV$_{\mathrm{II}}$ (4) and the dhToda (6), as $\delta^{(n)} \to \infty$. Let $Q_i^{(0)} = 5$, $i = 1,2,\ldots,12$ and $E_i^{(0)} = 2$, $i = 1,2,3$ with $M = 3$ and $m = 4$ in the dhToda (6). Moreover, let

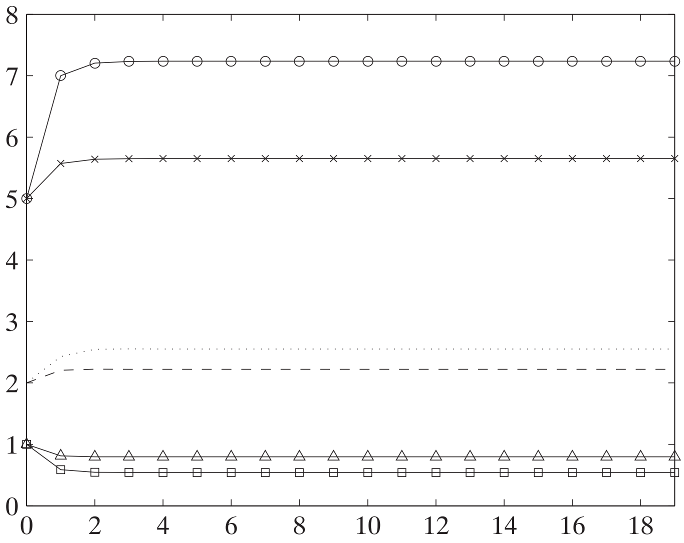

Figure 1: A graph of the iteration number n (x-axis) and the values of $\omega_1^{(n)}, \omega_2^{(n)}, \omega_4^{(n)}, \omega_5^{(n)}, \omega_7^{(n)}$ and $\omega_8^{(n)}$ (y-axis). $\circ$: $\omega_1^{(n)}$, $\times$: $\omega_2^{(n)}$, dotted line: $\omega_4^{(n)}$, dashed line: $\omega_5^{(n)}$, $\square : \omega_7^{(n)}$ and $\triangle : \omega_8^{(n)}$.

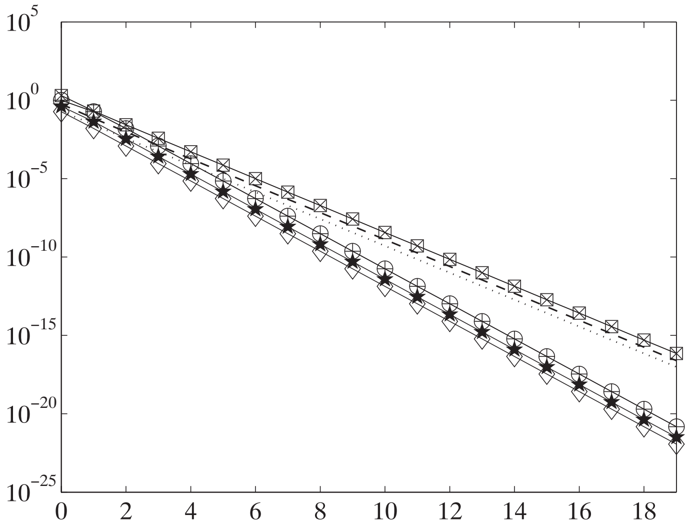

Figure 2: A graph of the iteration number n (x-axis) and the values of $\omega_3^{(n)}, \omega_6^{(n)}$ and $\gamma_k^{(n)}$ for $k = 1,2,3,4,5,6$ (y-axis). $\times : \omega_3^{(n)}$, $\circ : \omega_6^{(n)}$, $\square : \gamma_1^{(n)}$, dashed line: $\gamma_2^{(n)}$, dotted line: $\gamma_3^{(n)}$, $+ : \gamma_4^{(n)}$, $\star : \gamma_5^{(n)}$ and $\diamond : \gamma_6^{(n)}$.

$\omega_{M_i+j}^{(0)} = 5$, $i = 1,2,3,4$, $j = 0,1,2$ and $\omega_{M_i+M}^{(0)} = 2$, $i = 1,2,3$ with $M = 3$ and $m = 4$ in the qd-type dhLV$_{\mathrm{II}}$ (13). We consider two cases where $\delta^{(\ell)} = 0.5$ and $\delta^{(\ell)} = 10^{12}$ for $\ell = 0,1,\ldots$. In Tables 1 and 2, the 1st, the 2nd and the 3rd columns denote the iteration number ℓ, the dhToda variables and the qd-type dhLV$_{\mathrm{II}}$ ones, respectively. Table 1 illustrates that, in the case where $\delta^{(\ell)} = 0.5$, $\ell = 0,1,2,3,4,5$, the dhToda and dhLV$_{\mathrm{II}}$ variables coincide with each other in only a few digits. On the other hand, Table 2 shows that there are little difference

Table 1: Values of $Q_1^{(\ell M)}, \omega_{M_1}^{(\ell)}$ and $E_1^{(\ell M)}, \gamma_{M_1}^{(\ell)}$ in the case where $\delta^{(\ell)} = 0.5$ for $\ell = 0, 1, \ldots, 5$ and 50.

ℓ	$Q_1^{(\ell M)}$	$\omega_{M_1}^{(\ell)}$
0	5.00000000000	5.00000000000
1	7.00000000000	6.96850393700
2	7.90909090909	7.89572522147
3	8.34969325153	8.34251316996
4	8.60502692998	8.60022866410
5	8.76967867989	8.76612205167
⋮	⋮	⋮
50	9.04510897048	9.04510897048

ℓ	$E_1^{(\ell M)}$	$\gamma_{M_1}^{(\ell)}$
0	2.00000000000	1.96850393700
1	0.909090909090	0.927221284472
2	0.440602342442	0.446787948489
3	0.255333678448	0.257715494134
4	0.164651749916	0.165893387566
5	0.107582241979	0.108492550373
⋮	⋮	⋮
50	1.225191026031E−12	1.407359466438E−12

Table 2: Values of $Q_1^{(\ell M)}, \omega_{M_1}^{(\ell)}$ and $E_1^{(\ell M)}, \gamma_{M_1}^{(\ell)}$ in the case where $\delta^{(\ell)} = 10^{12}$ for $\ell = 0, 1, \ldots, 5$ and 50.

ℓ	$Q_1^{(\ell M)}$	$\omega_{M_1}^{(\ell)}$
0	5.00000000000	5.00000000000
1	7.00000000000	6.99999999999
2	7.90909090909	7.90909090909
3	8.34969325153	8.34969325153
4	8.60502692998	8.60502692998
5	8.76967867989	8.76967867989
⋮	⋮	⋮
50	9.04510897048	9.04510897048

ℓ	$E_1^{(\ell M)}$	$\gamma_{M_1}^{(\ell)}$
0	2.00000000000	1.99999999999
1	0.909090909090	0.909090909090
2	0.440602342442	0.440602342442
3	0.255333678448	0.255333678448
4	0.164651749916	0.164651749916
5	0.107582241979	0.107582241979
⋮	⋮	⋮
50	1.225191026031E−12	1.225191026031E−12

between $Q_1^{(\ell M)}, E_1^{(\ell M)}$ and $\omega_{M_1}^{(\ell)}, \gamma_{M_1}^{(\ell)}$, respectively. Tables 1 and 2 show that Theorem 5 approximately holds for a sufficiently large $\delta^{(\ell)}$.

In [9], some of the authors have proposed an algorithm for computing eigenvalues of totally nonnegative matrices, for which all the minors are nonnegative. So, from the Bäcklund transformation among the dhLV$_{\rm II}$ (4), the qd-type dhLV$_{\rm II}$ (13) and the dhToda (6) shown in Section 4, it is easily expected that the dhLV$_{\rm II}$ (4) and the qd-type dhLV$_{\rm II}$ (13) are applicable for computing the eigenvalues of a totally nonnegative matrix. In particular, $\lim_{n\to\infty} \prod_{p=0}^{M-1} \omega_{M_i+p}^{(n)}$, $^\forall i \in \Phi_4$ give the eigenvalues of the totally nonnegative matrix $A^{(0)}$.

6. Conclusion

In this paper, we first introduce the qd-type dhLV$_{\rm II}$ and show the positivity of its variables. We also give a new Lax representation for the dhLV$_{\rm II}$. As $\delta^{(n)} \to \infty$, it is observed that the Lax representation for the dhLV$_{\rm II}$ is related to the LR transformation for a band matrix. In other words, the time evolution of the dhLV$_{\rm II}$ with $\delta^{(n)} \to \infty$ corresponds to the LR transformation. We next explain how to associate the dhToda with the LR transformation. Remarkably, the dhLV$_{\rm II}$ with $\delta^{(n)} \to \infty$ is associated with the same form of LR transformation associated with the dhToda. By identifying two these LR transformations, we finally obtain a Bäcklund transformation between the dhLV$_{\rm II}$ and the dhToda. Additionally, through considering a Bäcklund transformation between the dhLV$_{\rm I}$ and the dhToda in [10], we establish a Bäcklund transformation between the dhLV$_{\rm I}$ and the dhLV$_{\rm II}$ for the case of $\delta^{(n)} \to \infty$. We therefore have Bäcklund transformations among the dhLV$_{\rm I}$, the dhLV$_{\rm II}$ and the dhToda. With the help of the Bäcklund transformation between the dhLV$_{\rm II}$ and the dhToda, we investigate the asymptotic convergence of the qd-type dhLV$_{\rm II}$ variables as $n \to \infty$ through that of the dhToda. We finally give some numerical examples which demonstrate our theoretical results.

We give a comment that the dhLV$_{\rm II}$ and the qd-type dhLV$_{\rm II}$ are applicable for computing eigenvalues of a totally nonnegative matrix. A future work is to derive the Bäcklund transformations among the dhLV$_{\rm I}$, the dhLV$_{\rm II}$ and the dhToda in the case where $\delta^{(n)}$ is finite.

Acknowledgements

This work is partially supported by Grants-in-Aid for Young Scientist (B) No.20740064, Scientific Research (C) No.23540163 and Scientific Research (A) No.20246027 of Japan Society for the Promotion of Science (JSPS).

References

[1] K. Narita: Soliton solution to extended Volterra equations, *J. Phys. Soc. Jpn.* **51** (1982) 1682–1685.

[2] O. I. Bogoyavlensky: Some constructions of integrable dynamical systems, *Math. USSR Izv* **31** (1998) 47–75.

[3] Y. Itoh: Integrals of a Lotka-Volterra system of odd number of variables, *Prog. Theor. Phys.* **78** (1987) 507–510.

[4] V. G. Papageorgiou, and F. W. Nijhoff: On some integrable discrete-time systems associated with the Bogoyavlensky lattices, *Physica A* **228** (1996) 172–188.

[5] R. Hirota and S. Tsujimoto: Conserved quantities of discrete Lotka-Volterra equation, *RIMS Kokyuroku* **868** (1994) 31–38.

[6] Y. B. Suris: Integrable discretizations of the Bogoyavlensky lattices, *J. Math. Phys.* **37** (1996) 3982–3996.

[7] A. Fukuda, E. Ishiwata, M. Iwasaki, and Y. Nakamura: The discrete hungry Lotka-Volterra system and a new algorithm for computing matrix eigenvalues, *Inverse Problems* **25** (2009) 015007.

[8] T. Tokihiro, A. Nagai, and J. Satsuma: Proof of solitonical nature of box and ball systems by means of inverse ultra-discretization, *Inverse Problems* **15** (1999) 1639–1662.

[9] A. Fukuda, E. Ishiwata, Y. Yamamoto, M. Iwasaki, and Y. Nakamura: The integrable discrete hungry systems and their related matrix eigenvalues, *Tech. Rep. Kyoto Univ.* **2010-016**, (2010).

[10] A. Fukuda, Y. Yamamoto, M. Iwasaki, E. Ishiwata, and Y. Nakamura: A Bäcklund transformation between two integrable discrete hungry systems, *Phys. Lett. A* **375** (2011) 303–308.

[11] C. Rogers and W. K. Schief: Bäcklund and Darboux Transformations, Geometry and Mordern Applications in Soliton Theory, Cambridge University Press, 2002.

[12] H. Rutishauser: Solution of eigenvalue problems with the LR-transformation, *Natl. Bur. Stand. Appl. Math.* **49** (1958) 47–81.

[13] A. Fukuda, E. Ishiwata, M. Iwasaki, and Y. Nakamura: On the qd-type discrete hungry Lotka-Volterra system and its application to the matrix eigenvalue algorithm, *JSIAM Lett.* **1** (2009) 36–39.

Yosuke Hama
Graduate School of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: j1410621(at)ed.kagu.tus.ac.jp

Akiko Fukuda
Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: afukuda(at)rs.tus.ac.jp

Yusaku Yamamoto
Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan
E-mail: yamamoto(at)cs.kobe-u.ac.jp

Masashi Iwasaki
Department of Informatics and Environmental Science, Kyoto Prefectural University, 1-5 Nakaragi-cho, Shimogamo, Sakyo-ku, Kyoto 606-8522, Japan
E-mail: imasa(at)kpu.ac.jp

Emiko Ishiwata
Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail: ishiwata(at)rs.tus.ac.jp

Yoshimasa Nakamura
Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
E-mail: ynaka(at)i.kyoto-u.ac.jp