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Piecewise truncated conical minimal surfaces and the Gauss hypergeometric functions

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Abstract. The catenary is the curve which a hanging chain forms, that is, mathematically, the graph of the function $t \mapsto c \cosh \frac{t}{c}$ for a constant $c > 0$. The study of catenaries is applied to the design of arches and suspension bridges. The surface of revolution generated by a catenary is called a catenoid. It is well-known that a catenoid is a minimal surface and the shape which a soap film between two parallel circles forms. In this article, we consider the approximation of a catenoid by combinations of some truncated cones keeping the minimality in a certain sense. In investigating the *minimal* combinations, the theory of the Gauss hypergeometric functions plays an important role.

Keywords. hypergeometric function, truncated cone, catenoid

1. Introduction

It is interesting to approximate a surface by *good* surfaces from an industrial point of view. In this article, we consider the approximation of a catenoid bounded by two circles of the same radii by a sequence of piecewise truncated conical minimal surfaces.

Throughout this article, a *truncated cone* means a right circular cone with its apex cut off by a plane parallel to the cone base.

For $x_0, x_1 > 0$ and $\ell > 0$, let $D_{1,\ell}(x_0, x_1)$ be the truncated cone such that the radii of two circles of it are x_0 and x_1, and its height is ℓ. Here, we do not consider the interior of the two circles of radii x_0 and x_1 of $D_{1,\ell}(x_0, x_1)$. Putting

$$S_{1,\ell}(x_0, x_1) := (x_0 + x_1)\sqrt{(x_1 - x_0)^2 + \ell^2},$$

the area of $D_{1,\ell}(x_0, x_1)$ is equal to $\pi \cdot S_{1,\ell}(x_0, x_1)$.

For $x_0, x_1, x_2 > 0$ and $\ell > 0$, let $D_{2,\ell}(x_0, x_1, x_2)$ be the figure consisting of the union of $D_{1,\ell}(x_0, x_1)$ and $D_{1,\ell}(x_1, x_2)$ attached along the circle of radius x_1. Similarly, for $n \geq 3$, we define $D_{n,\ell}(x_0, x_1, \ldots, x_{n-1}, x_n)$ inductively as the union of $D_{n-1,\ell}(x_0, x_1, \ldots, x_{n-1})$ and $D_{1,\ell}(x_{n-1}, x_n)$ attached along the circle of radius x_{n-1}. $D_{n,\ell}(x_0, x_2, \ldots, x_n)$ consists of n truncated cones and is called a *piecewise truncated conical surface* with length $(n; \ell)$ or simply a PTC surface with L-$(n; \ell)$ by definition. $D_{n,\ell}(x_0, x_1, \ldots, x_n)$ has the boundary consisting of two circles of radii x_0 and x_n, and its area is equal to

$$\pi \sum_{i=1}^{n} S_{1,\ell}(x_{i-1}, x_i).$$

We put

$$S_{n.\ell}(x_0, x_1, \ldots, x_n) := \sum_{i=1}^{n} S_{1,\ell}(x_{i-1}, x_i).$$

For arbitrary fixed $a, b > 0$ and $n \in \mathbb{N}$,

$$D_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b)$$

is called a PTC surface with boundary condition (a, b) and length $(n+2; \ell)$ or simply BCL-$(a, b; n+2; \ell)$. A PTC surface $D_{n+2,\ell}(a, x_0^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}, b)$ with BCL-$(a, b; n+2; \ell)$ is said to be *minimal* by definition if $\left(x_0^{(0)}, x_1^{(0)}, \ldots, x_s^{(0)}\right)$ is a critical point of the function

$$(x_0, x_1, \ldots, x_n) \mapsto S_{n+2,\ell}(a, x_0, x_1, \ldots, x_n, b).$$

Moreover a PTC minimal surface

$$D_{n+2,\ell}(a, x_0^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}, b)$$

with BCL-$(a, b; n+2; \ell)$ is said to be *stable* if and only if the Hessian matrix of the above function is positive definite at $\left(x_0^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}\right)$.

Putting

$$\begin{aligned} &2D_{n,\ell}(x_0, x_1, \ldots, x_n) \\ &:= D_{2n,\ell}(x_n, x_{n-1}, \ldots, x_1, x_0, x_1, \ldots, x_{n-1}, x_n), \end{aligned}$$

$2D_{n,\ell}(x_0, x_1, \ldots, x_{n-1}, a)$ is a PTC surface with BCL-$(a, a; 2n; \ell)$ for arbitrary fixed $a > 0$. We put

$$\tilde{S}_{a,n,\ell}(x_0, x_1, \ldots, x_{n-1}) := S_{n,\ell}(x_0, x_1, \ldots, x_{n-1}, a).$$

If $\left(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}\right)$ is a critical point of $\tilde{S}_{a,n,\ell}$, then $2D_{n,\ell}(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}, a)$ is minimal with BCL-$(a, a; n+2; \ell)$. Moreover, if the Hessian matrix of $\tilde{S}_{a,n,\ell}$ is positive definite there, then $2D_{n,\ell}(x_0^{(0)}, x_1^{(0)}, \ldots, x_{n-1}^{(0)}, a)$ is stable.

Now, we introduce the main results.

Theorem 1. *For $n \in N$ and $\ell > 0$, there are an explicit function $g_{n,\ell}(x)$ and a positive number $\eta_{n,\ell} > 0$ satisfying the following:*

(1) *If $a > \eta_{n,\ell}$, then the equation $g_{n,\ell}(x) - a = 0$ has two positive solutions $x^{\pm}_{a,n,\ell}$ with $x^-_{a,n,\ell} < x^+_{a,n,\ell}$.*

(2) *We see that*

$$2D_{n,\ell}(x_{a,n,\ell^{\pm}}, g_{1,\ell}(x^{\pm}_{a,n,\ell}), \ldots, g_{n-1,\ell}(x^{\pm}_{a,n,\ell}), a)$$

are PTC minimal surfaces with BCL-$(a, a; 2n; \ell)$.

Moreover,

$$g_{n,\ell}(x) = xT_n\left(1 + \frac{\ell^2}{2x^2}\right),$$

where T_n is the (first kind) Chebyshev polynomial.

Theorem 2. *Under the same situation as Theorem 1,*

$$2D_{n,\ell}(x_{a,n,\ell^{+}}, g_{1,\ell}(x^{+}_{a,n,\ell}), \ldots, g_{n-1,\ell}(x^{+}_{a,n,\ell}), a)$$

is stable.

2. The cases $n = 1, 2, 3$

In this section we see Theorems 1 and 2 in the cases where $n = 1, 2, 3$.

2.1. The case $n = 1$

For $a > 0$ and $\ell > 0$, we consider the critical points of the function

$$\tilde{S}_{a,1,\ell}(x_0) := S_{1,\ell}(x_0, a) = (x_0 + a)\sqrt{(a - x_0)^2 + l^2}.$$

Since

$$\frac{d\tilde{S}_{a,1,\ell}}{dx_0} = \frac{2x_0^2 - 2ax_0 + \ell^2}{\sqrt{(a - x_0)^2 + \ell^2}},$$

if $a > \sqrt{2}\ell =: \eta_{1,\ell}$, then there are two critical points

$$(a \pm \sqrt{a^2 - 2\ell^2})/2$$

of $\tilde{S}_{a,1,\ell}(x_0)$. We put

$$x^+_{a,1,\ell} := \frac{a + \sqrt{a^2 - 2\ell^2}}{2},\ x^-_{a,1,\ell} := \frac{a - \sqrt{a^2 - 2\ell^2}}{2}.$$

Then, since

$$a = x^+_{a,1,\ell} + \frac{\ell^2}{2x^+_{a,1,\ell}} = x^-_{a,1,\ell} + \frac{\ell^2}{2x^-_{a,1,\ell}},$$

if we put

$$g_{1,\ell}(x) := x + \frac{\ell^2}{2x}$$

for $x > 0$, then

$$\{x^{\pm}_{a,1,\ell}\} = g^{-1}_{1,\ell}(a)$$

for $a > \eta_{1,\ell}$. In other words, a positive number x is a critical point of $\tilde{S}_{g_{1,\ell}(x),1,\ell}(x_0)$.

We remark that $g_1(x)$ takes the minimum $\eta_{1,\ell}$ at $x = \ell/\sqrt{2}$, that is, $g'_{1,\ell}(\ell/\sqrt{2}) = 0$ and $g_{1,\ell}(\ell/\sqrt{2}) = \eta_{1,\ell}$. Thus, putting $\xi_{1,\ell} := \ell/\sqrt{2}$,

$$x^+_{a,1,\ell} > \xi_{1,\ell} > x^-_{a,1,\ell}$$

for $a > \eta_{1,\ell}$.

Moreover, if we put for $x > 0$, $a := g_{1,\ell}(x)$,

$$H_{1,\ell}(x) := \frac{d^2\tilde{S}_{a,1,\ell}}{dx_0^2}(x),$$

and

$$\det H_{1,\ell}(x) := H_{1,\ell}(x)$$

itself, then

$$\begin{aligned}
\det H_1(x) &= \frac{\ell^2(3x - a) + 2(x - a)^3}{((a - x)^2 + \ell^2)^{3/2}} \\
&= \frac{\ell^2(3x - (x + \frac{\ell^2}{2x})) + 2(x - (x + \frac{\ell^2}{2x}))^3}{((a - x)^2 + \ell^2)^{3/2}} \\
&= \frac{\ell^2(2x^2 - \ell^2)(4x^2 + 1)}{4x^3((a - x)^2 + \ell^2)^{3/2}} \\
&= \frac{\ell^2(4x^2 + 1)g'_{1,\ell}(x)}{2x((a - x)^2 + \ell^2)^{3/2}} \\
&= \frac{4x^2 g'_{1,\ell}(x)}{\ell(4x^2 + \ell^2)^{1/2}}.
\end{aligned}$$

Together with the behavior of $g'_{1,\ell}(x)$, this formula means that for $a > \eta_{1,\ell}$, $\tilde{S}_{a,1,\ell}(x_0)$ takes the local minimum at $x^+_{a,1,\ell}$ because $g_{1,\ell}(x^+_{a,1,\ell}) > 0$.

2.2. The case $n = 2$

For $a > 0$ and $\ell > 0$, we consider the critical points of the function $\tilde{S}_{a,2,\ell}(x_0, x_1) := S_{1,\ell}(x_0, x_1) + S_{1,\ell}(x_1, a)$, that is, we consider a point (x_0, x_1) satisfying

$$\frac{\partial\tilde{S}_{a,2,\ell}}{\partial x_0}(x_0, x_1) = \frac{\partial\tilde{S}_{a,2,\ell}}{\partial x_1}(x_0, x_1) = 0.$$

By the case where $n = 1$ and the formula

$$\frac{\partial\tilde{S}_{a,2,\ell}}{\partial x_0}(x_0, x_1) = 0,$$

we see that $x_1 = g_{1,\ell}(x_0)$. Moreover,

$$\begin{aligned}
0 &= \frac{\partial\tilde{S}_{a,2,\ell}}{\partial x_1}(x_0, x_1) \\
&= \frac{2x_1^2 - 2x_0x_1 + \ell^2}{\sqrt{(x_1 - x_0)^2 + \ell^2}} + \frac{2x_1^2 - 2ax_1 + \ell^2}{\sqrt{(a - x_1)^2 + \ell^2}} \qquad (2.1)
\end{aligned}$$

implies that

$$\begin{aligned}0 &= (2x_1^2 - 2x_0x_1 + \ell^2)^2((a - x_1)^2 + \ell^2)\\ &\quad - (2x_1^2 - 2ax_1 + \ell^2)^2((x_1 - x_0)^2 + \ell^2)\\ &= \ell^2(a - x_0)(4x_1^3 - 4ax_0x_1 + 2\ell^2x_1 + \ell^2x_0 + a\ell^2).\end{aligned}$$

Here, if $a = x_0$, then Formula (2.1) does not hold and so we have

$$(\ell^2 - 4x_0x_1)a + 4x_1^3 + 2\ell^2x_1 + \ell^2x_0 = 0,$$

that is,

$$\begin{aligned}a &= \frac{4x_1^3 + 2\ell^2x_1 + \ell^2x_0}{4x_0x_1 - \ell^2}\\ &= \frac{4\left(g_{1,\ell}(x_0)\right)^3 + 2\ell^2g_{1,\ell}(x_0) + \ell^2x_0}{4x_0g_{1,\ell}(x_0) - \ell^2}\\ &= x_0 + \frac{2\ell^2}{x_0} + \frac{\ell^4}{2x_0^3},\end{aligned}$$

where we can check that in this case, Formula (2.1) holds.

If we put

$$g_{2,\ell}(x) := x + \frac{2\ell^2}{x} + \frac{\ell^4}{2x^3},$$

then $g_{2,\ell}(x)$ is positive, convex in $(0, \infty)$, and

$$\lim_{x\to 0} g_{2,\ell}(x) = \lim_{x\to\infty} g_{2,\ell}(x) = \infty$$

and thus, it takes the unique minimal value $\eta_{2,\ell} > 0$ at a point $\xi_{2,\ell} > 0$. Hence, if $a > \eta_{2,\ell}$, then there are two solutions $x^{\pm}_{a,2,\ell}$ of $g_{2,\ell}(x) = a$, where $x^-_{a,2,\ell} < \xi_{2,\ell} < x^+_{a,2,\ell}$. Consequently, if $a > \eta_{2,\ell}$, then there are two critical points $\left(x^{\pm}_{a,2,\ell}, g_{1,\ell}\left(x^{\pm}_{a,2,\ell}\right)\right)$ of $\tilde{S}_{a,2,\ell}(x_0, x_1)$; if $a = \eta_{2,\ell}$, only one critical point $(\xi_{2,\ell}, g_{1,\ell}(\xi_{2,\ell}))$; and if $a < \eta_{2,\ell}$, there is no critical point.

We should remark that since $g'_{1,\ell}(x) > g'_{2,\ell}(x)$ for $x > 0$, $\xi_{1,\ell} < \xi_{2,\ell}$. By numeric calculations we see that $\xi_{2,\ell} \approx 1.6066\ell$ and $\eta_{2,\ell} \approx 2.9720\ell$.

Seeing the above argument in terms of x_0, we have that for $x > 0$, if we put $a := g_{2,\ell}(x)$, then $(x, g_{1,\ell}(x))$ is a critical point of $\tilde{S}_{a,\ell}(x_0, x_1)$.

Next, for $x > 0$, putting $a = g_{2,\ell}(x)$, we investigate the Hessian matrix

$$H_{2,\ell}(x) := \begin{pmatrix} \frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) & \frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0\partial x_1}(x, g_{1,\ell}(x))\\ \frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0\partial x_1}(x, g_{1,\ell}(x)) & \frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_1^2}(x, g_{1,\ell}(x))\end{pmatrix}$$

of $\tilde{S}_{a,2,\ell}(x_0, x_1)$ at $(x, g_{1,\ell}(x))$.

From this point on, we put, for $\ell > 0$ and $s, t > 0$,

$$S_\ell(s, t) := (s + t)\sqrt{(t - s)^2 + \ell^2} \quad (= S_{1,\ell}(s, t)).$$

Then, we see

$$\begin{aligned}\frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) &= \frac{\partial^2 S_\ell}{\partial s^2}(x, g_{1,\ell}(x)),\\ \frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0\partial x_1}(x, g_{1,\ell}(x)) &= \frac{\partial^2 S_\ell}{\partial s\partial t}(x, g_{1,\ell}(x)),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_1^2}(x, g_{1,\ell}(x)) =& \frac{\partial^2 S_\ell}{\partial t^2}(x, g_{1,\ell}(x))\\ &+ \frac{\partial^2 S_\ell}{\partial s^2}(g_{1,\ell}(x), g_{2,\ell}(x)).\end{aligned}$$

Then by a direct but long calculation using

$$(g_{2,\ell}(x) - g_{1,\ell}(x))^2 + \ell^2 = \frac{(x^2 + \ell^2)^2}{x^4}\left((g_{1,\ell}(x) - x)^2 + \ell^2\right),$$

we see that the determinant $\det H_{2,\ell}(x)$ of $H_{2,\ell}(x)$ satisfies

$$\begin{aligned}\det H_{2,\ell}(x) &= \frac{16x^8\left(1 - \frac{2\ell^2}{x^2} - \frac{3\ell^4}{2x^4}\right)}{\ell^2(x^2 + \ell^2)^2(4x^2 + l^2)}\\ &= \frac{16x^8 g'_{2,\ell}(x)}{\ell^2(x^2 + \ell^2)^2(4x^2 + l^2)}.\end{aligned}$$

Thus, if $x > \xi_{2,\ell}$, then $\det H_{2,\ell}(x) > 0$. Moreover $\xi_{2,\ell} > \xi_{1,\ell}$ implies that if $x > \xi_{2,\ell}$, then $x > \xi_{1,\ell}$ and

$$\frac{\partial^2\tilde{S}_{a,2,\ell}}{\partial x_0^2}(x, g_{1,\ell}(x)) = \frac{\partial^2 S_\ell}{\partial s^2}(x, g_{1,\ell}(x)) = H_{1,\ell}(x) > 0$$

from the case where $n = 1$. This implies that $H_{2,\ell}(x)$ is positive definite at $\left(x^+_{a,2,\ell}, g_{1,\ell}(x^+_{a,2,\ell})\right)$ if $a > \eta_{2,\ell}$ and

$$2D_{2,\ell}(x^+_{a,2,\ell}, g_{1,\ell}(x^+_{a,2,\ell}), a)$$

is a stable PTC minimal surface with BCL-$(a, a; 4; \ell)$.

2.3. The case $n = 3$

We consider the critical points of

$$\tilde{S}_{a,3,\ell}(x_0, x_1, x_2) := S_\ell(x_0, x_1) + S_\ell(x_1, x_2) + S_\ell(x_2, a)$$

for $a > 0$. If (x_0, x_1, x_2) is a critical point of $\tilde{S}_{a,3,\ell}$, then as in the case where $n = 2$, we have

$$\begin{aligned}x_1 &= g_{1,\ell}(x_0),\\ x_2 &= g_{2,\ell}(x_0),\end{aligned}$$

and

$$\begin{aligned}a &= \frac{4x_2^3 + 2\ell^2x_2 + \ell^2x_1}{4x_1x_2 - \ell^2}\\ &= \frac{4\left(g_{2,\ell}(x_0)\right)^3 + 2\ell^2g_{2,\ell}(x_0) + \ell^2g_{1,\ell}(x_0)}{4g_{1,\ell}(x_0)g_{2,\ell}(x_0) - \ell^2}\\ &= x_0 + \frac{9\ell^2}{2x_0} + \frac{3\ell^4}{x_0^3} + \frac{\ell^6}{2x_0^5}.\end{aligned}$$

Putting

$$g_{3,\ell}(x) := x + \frac{9\ell^2}{2x} + \frac{3\ell^4}{x^3} + \frac{\ell^6}{2x^5},$$

similarly as in the case $n = 2$, we see that there is $\xi_{3,\ell} > 0$ with $g'_{3,\ell}(\xi_{3,\ell}) = 0$ such that if $a > \eta_{3,\ell} :=$

$g_{3,\ell}(\xi_{3,\ell})$, then the equation $g_{3,\ell}(x) = a$ has two solutions $x^{\pm}_{a,3,\ell}$ with $x^{+}_{a,3,\ell} > \xi_{3,\ell} > x^{-}_{a,3,\ell}$. Moreover $\left(x^{\pm}_{a,3,\ell}, g_{1,\ell}(x^{\pm}_{a,3,\ell}), g_{2,\ell}(x^{\pm}_{a,3,\ell})\right)$ are the critical points of $\tilde{S}_{a,3,\ell}(x_0, x_1, x_2)$. The same argument as in the case $n = 2$ implies $\xi_{3,\ell} > \xi_{2,\ell}$.

We define $H_{3,\ell}(x)$ for $x > 0$ as the Hessian matrix of $\tilde{S}_{a,3,\ell}$ at $(x, g_{1,\ell}(x), g_{2,\ell}(x))$, where $a := g_{3,\ell}(x)$. Then

$$\frac{\partial^2 \tilde{S}_{a,3,\ell}}{\partial x_0 \partial x_2} = 0$$

implies

$$\det H_{3,\ell}(x) = \frac{\partial^2 \tilde{S}_{a,3,\ell}}{\partial x_2^2} \times \det H_{2,\ell}(x) - \left(\frac{\partial^2 \tilde{S}_{a,3,\ell}}{\partial x_1 \partial x_2}\right)^2 \times \det H_{1,\ell}(x).$$

Making a long calculation (with the help of a computer), we see that

$$\det H_{3,\ell}(x) = \frac{64 x^{18} g'_{3,\ell}(x)}{\ell^3 (x^2+\ell^2)^2 (x^4 + 3\ell^2 x^2 + \ell^4)^2 (4x^2+\ell^2)^{3/2}}.$$

Thus, by $\xi_{3,\ell} > \xi_{2,\ell} > \xi_{1,\ell}$, if $x > \xi_{3,\ell}$, then $\det H_{3,\ell}(x) > 0, \det H_{2,\ell}(x) > 0$, and $\det H_{1,\ell}(x) > 0$ and as is well-known in linear algebra, this implies $H_{3,\ell}$ is positive definite. (See Lemma 3 described in Section 6.) Consequently,

$$2D_{3,\ell}\left(x^{+}_{a,3,\ell}, g_{1,\ell}(x^{+}_{a,3,\ell}), g_{2,\ell}(x^{+}_{a,3,\ell}), a\right)$$

is stable.

The calculation of the determinant of $H_n(x)$ is mentioned later.

Repeating the above argument, we see that $g_{4,\ell}(x)$ and $g_{5,\ell}(x)$ should be defined as

$$\begin{aligned} g_{4,\ell}(x) :&= \frac{4\left(g_{3,\ell}(x)\right)^3 + 2\ell^2 g_{3,\ell}(x) + \ell^2 g_{2,\ell}(x)}{4 g_{2,\ell}(x) g_{3,\ell}(x) - \ell^2} \\ &= x + \frac{8\ell^2}{x} + \frac{10\ell^4}{x^3} + \frac{4\ell^6}{x^5} + \frac{\ell^8}{2x^7}, \end{aligned}$$

$$\begin{aligned} g_{5,\ell}(x) :&= \frac{4\left(g_{4,\ell}(x)\right)^3 + 2\ell^2 g_{4,\ell}(x) + \ell^2 g_{3,\ell}(x)}{4 g_{3,\ell}(x) g_{4,\ell}(x) - \ell^2} \\ &= x + \frac{25\ell^2}{2x} + \frac{25\ell^4}{x^3} + \frac{35\ell^6}{2x^5} + \frac{5\ell^8}{x^7} + \frac{l^{10}}{2x^9}, \end{aligned}$$

and in general,

$$g_{n,\ell}(x) := \frac{4\left(g_{n-1,\ell}(x)\right)^3 + 2\ell^2 g_{n-1,\ell}(x) + \ell^2 g_{n-2,\ell}(x)}{4 g_{n-2,\ell}(x) g_{n-1,\ell}(x) - \ell^2}$$

for $n \geq 2$, here $g_{0,\ell}(x) := x$.

3. Catenoids and approximations of them

We put for $c > 0$,

$$C_c(t) := c \cosh\left(\frac{t}{c}\right).$$

The curve $(t, C_c(t))$ is called a catenary. The function $c \mapsto c\cosh\left(\frac{1}{c}\right)$ is positive, convex and takes the unique minimum $\eta_\infty := 1.5088\cdots$ at $c = 0.83355\cdots =: \xi_\infty$. Thus, if $a > \eta_\infty$, there are two positive numbers $c^{\pm}_a$ with $c^{-}_a < \xi_\infty < c^{+}_a$ such that $c^{\pm}_a \cosh\left(\frac{1}{c^{\pm}_a}\right) = a$.

The surface $R(C_c) := (t, C_c(t)\cos\theta, C_c(t)\sin\theta)$ is called a catenoid, which is known as a minimal surface of revolution, where "minimal" means "of mean curvature 0". Let $C_{c,1}$ be $C_c|_{(-1,1)}$. For $a > \eta_\infty$, $R\left(C_{c^{\pm}_a,1}\right)$ have the same boundary. The area of $R\left(C_{c^{+}_a,1}\right)$ is minimal in the set of surfaces having the same boundary and that of $R\left(C_{c^{-}_a,1}\right)$ is not.

In the view of the previous section, if $a > \eta_\infty$, the sequence

$$\begin{aligned} &2D_{1,1}\left(x^{\pm}_{a,1,1}, a\right), \\ &2D_{2,\frac{1}{2}}\left(x^{\pm}_{a,2,\frac{1}{2}}, g_{1,\frac{1}{2}}\left(x^{\pm}_{a,2,\frac{1}{2}}\right), a\right), \\ &2D_{3,\frac{1}{3}}\left(x^{\pm}_{a,3,\frac{1}{3}}, g_{1,\frac{1}{3}}\left(x^{\pm}_{a,3,\frac{1}{3}}\right), g_{2,\frac{1}{3}}\left(x^{\pm}_{a,3,\frac{1}{3}}\right), a\right), \\ &\qquad\vdots \end{aligned}$$

might give an approximation of $R\left(C_{c^{\pm}_a,1}\right)$ as PTC minimal surfaces, where the formula $\eta_\infty > \eta_{n,\frac{1}{n}}$ is proved later.

For example, if $a = 2$, then

$$\begin{aligned} x^{+}_{2,1,1} &= 1.707\cdots, \\ x^{+}_{2,2,\frac{1}{2}} &= 1.699\cdots, \quad g_{1,\frac{1}{2}}(1.699\cdots) = 1.772\cdots, \\ x^{+}_{2,3,\frac{1}{3}} &= 1.697\cdots, \quad g_{1,\frac{1}{3}}(1.697\cdots) = 1.730\cdots, \\ &\qquad g_{2,\frac{1}{3}}(1.697\cdots) = 1.830\cdots, \end{aligned} \tag{3.1}$$

and thus,

$$\begin{aligned} &2D_1(1.707\cdots, 2), \\ &2D_{\frac{1}{2}}(1.699\cdots, 1.772\cdots, 2), \\ &2D_{\frac{1}{3}}(1.697\cdots, 1.730\cdots, 1.830\cdots, 2), \\ &\qquad\vdots \end{aligned}$$

might give an approximate of $R\left(C_{c^{+}_2,1}\right)$ as PTC minimal surfaces, here

$$\begin{aligned} c^{+}_2 &= 1.696\cdots, \\ c^{+}_2 &= 1.696\cdots, \quad C_{c^{+}_2}(1/2) = 1.770\cdots, \\ c^{+}_2 &= 1.696\cdots, \quad C_{c^{+}_2}(1/3) = 1.729\cdots, \\ &\qquad C_{c^{+}_2}(2/3) = 1.829\cdots. \end{aligned}$$

(Compare with (3.1).)

Referring to the expansion

$$c\cosh(1/c) = c + \frac{1}{2c} + \frac{1}{4!c^3} + \frac{1}{6!c^5} + \frac{1}{8!c^7} + \cdots,$$

we change $g_{n,\frac{1}{n}}(x)$ for $n = 2, 3, 4$ as follows:

$$\begin{aligned}
g_{2,\frac{1}{2}}(x) &= x + \frac{1}{2x} + \frac{1}{32x^3} \\
&= x + \frac{1}{2x} + \frac{1\cdot 3}{2^2}\cdot\frac{1}{4!x^3} \\
&= x + \frac{1}{2x} + \frac{3!}{0!\cdot 2^3}\cdot\frac{1}{4!x^3}, \\
g_{3,\frac{1}{3}}(x) &= x + \frac{1}{2x} + \frac{1}{27x^3} + \frac{1}{1458x^5} \\
&= x + \frac{1}{2x} + \frac{2\cdot 4}{3^2}\cdot\frac{1}{4!x^3} + \frac{1\cdot 2\cdot 4\cdot 5}{3^4}\cdot\frac{1}{6!x^5} \\
&= x + \frac{1}{2x} + \frac{4!}{1!\cdot 3^3}\cdot\frac{1}{4!x^3} + \frac{5!}{0!\cdot 3^5}\cdot\frac{1}{6!x^5}, \\
g_{4,\frac{1}{4}}(x) &= x + \frac{1}{2x} + \frac{5}{128x^3} + \frac{1}{1024x^5} + \frac{1}{131072x^7} \\
&= x + \frac{1}{2x} + \frac{3\cdot 5}{4^2}\cdot\frac{1}{4!x^3} + \frac{2\cdot 3\cdot 5\cdot 6}{4^4}\cdot\frac{1}{6!x^5} \\
&\quad + \frac{1\cdot 2\cdot 3\cdot 5\cdot 6\cdot 7}{4^6}\cdot\frac{1}{8!x^7} \\
&= x + \frac{1}{2x} + \frac{5!}{2!\cdot 4^3}\cdot\frac{1}{4!x^3} + \frac{6!}{1!\cdot 4^5}\cdot\frac{1}{6!x^5} \\
&\quad + \frac{7!}{0!\cdot 4^7}\cdot\frac{1}{8!x^7}.
\end{aligned}$$

Thus, it is indicated that

$$g_{n,\frac{1}{n}}(x) = \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-k)!\cdot(2k)!\cdot n^{2k-1}\cdot x^{2k-1}}. \tag{3.2}$$

In fact, we prove this formula in the next section. Assuming this, we see the following remark.

Remark 1. We put

$$g_\infty(x) := x\cosh\frac{1}{x}.$$

Then, the coefficient of $\frac{1}{x^{2k-1}}$ of $g_{n,\frac{1}{n}}(x)$ is larger than that of $g_{n-1,\frac{1}{n-1}}(x)$ and smaller than that of $g_\infty(x)$ for $n \geq 2$ and $2 \leq k \leq n$. Thus, we see that $g_\infty(x) > g_{n,\frac{1}{n}}(x) > g_{n-1,\frac{1}{n-1}}(x)$ and $g'_\infty(x) < g'_{n,\frac{1}{n}}(x) < g'_{n-1,\frac{1}{n-1}}(x)$ for $x > 0$. Moreover $g_{n,\frac{1}{n}}(x) \to g_\infty(x)$ as $n \to \infty$. Consequently we have that if we let $\xi_{n,\frac{1}{n}}$ be the zero point of $g'_{n,\frac{1}{n}}(x)$ and put $\eta_{n,\frac{1}{n}} := g_{n,\frac{1}{n}}(\xi_{n,\frac{1}{n}})$, then

$$\begin{aligned}
\xi_{1,1} < \xi_{2,\frac{1}{2}} < \xi_{3,\frac{1}{3}} < \cdots < \xi_\infty \\
\eta_{1,1} < \eta_{2,\frac{1}{2}} < \eta_{3,\frac{1}{3}} < \cdots < \eta_\infty
\end{aligned}$$

and

$$\lim_{n\to\infty} \xi_{n,\frac{1}{n}} \to \xi_\infty, \quad \lim_{n\to\infty} \eta_{n,\frac{1}{n}} \to \eta_\infty$$

4. Proof of Theorem 1

As is seen in the previous section, Formula (3.2) is indicated.

For $m \in \mathbb{N}\cup\{0\}$ and $y \in \mathbb{R}$, let $(y)_m$ be the Pochhammer symbol, that is, $(y)_0 := 1$ and for $m \in \mathbb{N}$

$$(y)_m := \prod_{i=0}^{m-1}(y+i).$$

Then, we see that

$$\frac{(n+k-1)!}{(n-k)!} = \frac{(-1)^k\cdot(n)_k\cdot(-n)_k}{n},$$

$$(2k)! = k!\cdot 4^k\cdot(\frac{1}{2})_k,$$

and

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(n+k-1)!}{(n-k)!\cdot(2k)!\cdot n^{2k-1}\cdot x^{2k-1}} \\
&= x\sum_{k=0}^{n}\frac{(n)_k\cdot(-n)_k}{(1/2)_k\cdot k!}\cdot\left(-\frac{1}{4(nx)^2}\right)^k.
\end{aligned} \tag{4.1}$$

For $\ell, \ell' > 0$, $a > 0$, and $(x_0, x_1, \ldots, x_n)$,

$$D_{n+1,\ell}(x_0, x_1, \ldots, x_n, a)$$

and

$$D_{n+1,\ell'}((\ell'/\ell)x_0, (\ell'/\ell)x_1, \ldots, (\ell'/\ell)x_n, (\ell'/\ell)a)$$

are homethetic to each other. Consequently, we have

$$g_{n,\ell}(x) = \frac{\ell}{\ell'}g_{n,\ell'}\left(\frac{\ell'}{\ell}x\right),$$

and if $\ell' = \frac{1}{n}$, then

$$g_{n,\ell}(x) = n\ell\cdot g_{n,\frac{1}{n}}\left(\frac{x}{n\ell}\right).$$

Substituting $\frac{x}{n\ell}$ instead of x in Formula (4.1), we propose that

$$g_{n,\ell}(x) = x\sum_{k=0}^{n}\frac{(n)_k\cdot(-n)_k}{(1/2)_k\cdot k!}\cdot\left(-\left(\frac{\ell}{2x}\right)^2\right)^k.$$

For $\alpha, \beta, \gamma \in \mathbb{R}$, where $\gamma \neq 0, -1, -2, \ldots$, the series

$$F(\alpha, \beta, \gamma; z) := \sum_{k=0}^{\infty}\frac{(\alpha)_k\cdot(\beta)_k}{(\gamma)_k\cdot k!}z^k$$

is called a Gauss hypergeometric function.

Since $(-n)_k = 0$ for $k \geq n+1$, we see

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(n)_k\cdot(-n)_k}{(1/2)_k\cdot k!}\cdot\left(-\left(\frac{\ell}{2x}\right)^2\right)^k \\
&= F\left(n, -n, \frac{1}{2}; -\left(\frac{\ell}{2x}\right)^2\right)
\end{aligned}$$

for $x > 0$.

Let T_n for $n \in \mathbb{N} \cup \{0\}$ be the Chebyshev polynomial, that is,

$$T_0(z) := 1, \quad T_1(z) := z$$

and for $n \geq 2$,

$$T_n(z) := 2zT_{n-1}(z) - T_{n-2}(z).$$

Then, it is well-known that $F(n, -n, \frac{1}{2}; z) = T_n(1 - 2z)$ (See 15.4.3 in [1].). Moreover, it is also well-known that

$$\begin{aligned} &F(n, -n, \frac{1}{2}; -z^2) \\ &= \frac{1}{2}\left([(1+z^2)^{\frac{1}{2}} + z]^{2n} + [(1+z^2)^{\frac{1}{2}} - z]^{2n}\right). \end{aligned}$$

(See 15.1.11 in [1].)

Lemma 1. *For $n \geq 2$, we see that*

$$T_{n-1}^2(x) - T_n(x)T_{n-2}(x) = 1 - x^2.$$

Proof. In the case of $n = 2$, we obtain this by direct calculation. For $n \geq 3$, by the recursion of the Chebyshev polynomials,

$$\begin{aligned} &T_{n-1}^2(x) - T_n(x)T_{n-2}(x) \\ &= T_{n-1}^2(x) - (2xT_{n-1}(x) - T_{n-2}(x))\,T_{n-2}(x) \\ &= T_{n-2}^2(x) + T_{n-1}(x)\,(T_{n-1}(x) - 2xT_{n-2}(x)) \\ &= T_{n-2}^2(x) - T_{n-1}T_{n-3}(x) \\ &\vdots \\ &= T_1^2(x) - T_0(x)T_2(x) \\ &= 1 - x^2. \end{aligned}$$

□

Proof of Theorem 1. Recall that the recursion formula which $g_{n,\ell}(x)$ should satisfy is

$$g_{n,\ell}(x) := \frac{4\,(g_{n-1,\ell}(x))^3 + 2\ell^2 g_{n-1,\ell}(x) + \ell^2 g_{n-2,\ell}(x)}{4g_{n-2,\ell}(x)g_{n-1,\ell}(x) - \ell^2}$$

for $n \geq 2$. (See the last paragraph of Section 2.) Since

$$g_{0,\ell}(x) = xT_0\left(1 + \frac{\ell^2}{2x^2}\right)$$

and

$$g_{1,\ell}(x) = xT_1\left(1 + \frac{\ell^2}{2x^2}\right),$$

it suffices to prove that $xT_n\left(1 + \frac{\ell^2}{2x^2}\right)$ satisfies the same recursion for $n \geq 2$. Rearranging the recursion, the formula we should show is

$$\begin{aligned} 4x^2T_{n-1}(X)\left(T_{n-1}^2(X) - T_n(X)\,T_{n-2}(X)\right) \\ + \ell^2\,(T_n(X) + 2T_{n-1}(X) + T_{n-2}(X)) = 0, \end{aligned} \tag{4.2}$$

where $X = 1 + \frac{\ell^2}{2x^2}$. Lemma 1 implies

$$\begin{aligned} T_{n-1}^2(X) - T_n(X)\,T_{n-2}(X) &= 1 - X^2 \\ &= -(\frac{\ell^2}{x^2} + \frac{\ell^4}{4x^4}), \end{aligned}$$

and the left side of Formula (4.2) is equal to

$$\ell^2\,(T_n(X) - 2XT_{n-1}(X) + T_{n-2}(X)) = 0.$$

Given these facts, we obtain

$$g_{n,\ell}(x) = x\sum_{k=0}^{n} \frac{(n)_k \cdot (-n)_k}{(1/2)_k \cdot k!} \cdot \left(-\left(\frac{\ell}{2x}\right)^2\right)^k$$

or

$$g_{n,\ell}(x) = \sum_{k=0}^{n} \frac{n \cdot (n+k-1)! \cdot \ell^{2k}}{(n-k)! \cdot (2k)! \cdot x^{2k-1}}.$$

Since this function is positive and convex for $x > 0$, and

$$\lim_{x\to 0} g_{n,\ell}(x) = \lim_{x\to\infty} g_{n,\ell}(x) = \infty,$$

there is a unique zero point $\xi_{n,\ell}$ of $g'_{n,\ell}(x)$. Moreover, if we put $\eta_{n,\ell} := g_{n,\ell}(\xi_{n,\ell})$, then $\eta_{n,\ell}$ is the minimum of $g_{n,\ell}$.

The role of $\eta_{n,\ell}$ and the minimality of

$$2D_{n,\ell}(x_{a,n,\ell^\pm}, g_{1,\ell}(x^\pm_{a,n,\ell}), \ldots, g_{n-1,\ell}(x^\pm_{a,n,\ell}), a)$$

are obtained similarly as in the case $n = 1, 2, 3$. □

Remark 2. The coefficient of $\frac{1}{x^{2k-1}}$ of $g_{n,\ell}(x)$ is larger than that of $g_{n-1,\ell}$ for $2 \leq k \leq n$. Thus, $g_{n,\ell}(x) > g_{n-1,\ell}(x)$ and $g'_{n,\ell}(x) < g_{n-1,\ell}(x)$ for $x > 0$. This implies that

$$\eta_{1,\ell} < \eta_{2,\ell} < \cdots < \eta_{n,\ell} < \cdots$$

and

$$\xi_{1,\ell} < \xi_{2,\ell} < \cdots < \xi_{n,\ell} < \cdots.$$

As is seen in Remark 1, we have

$$\lim_{n\to\infty} \xi_{n,\frac{1}{n}} = 0.83355\cdots \quad \text{and} \quad \lim_{n\to\infty} \eta_{n,\frac{1}{n}} = 1.5088\cdots.$$

Thus, by using the fact that

$$D_{\ell,n}(x_0, x_1, \ldots, x_n)$$

is homothetic to

$$D_{\ell',n}(\frac{\ell'}{\ell}x_0, \frac{\ell'}{\ell}x_1, \ldots, \frac{\ell'}{\ell}x_n)$$

for $\ell, \ell' > 0$, we see that $\xi_{n,\ell} = (\ell/\ell')\xi_{n,\ell'}$ and $\eta_{n,\ell} = (\ell/\ell')\eta_{n,\ell'}$ and that

$$\lim_{n\to\infty} \frac{\xi_{n,\ell}}{n} = 0.83355\cdots \times \ell, \lim_{n\to\infty} \frac{\eta_{n,\ell}}{n} = 1.5088\cdots \times \ell.$$

5. The Hessian matrices

The purpose of this section is to investigate the Hessian matrix of the function

$$\tilde{S}_{a,n,\ell}(x_0, x_1, x_2, \ldots, x_{n-1})$$

at

$$\left(g_0(x^+_{a,n,\ell}), g_1(x^+_{a,n,\ell}), g_2(x^+_{a,n,\ell}), \ldots, g_{n-1}(x^+_{a,n,\ell}))\right),$$

where we should remark that $g_n(x^+_{a,n,\ell}) = a$. For investigating the positive definiteness of the matrix, we may assume that $\ell = 1$ without loss of generality. Thus, we put $S(s,t) := S_1(s,t) = (s+t)\sqrt{(t-s)^2+1}$, $g_k(x) := g_{k,1}(x)$, and $x^+_{a,n} := x^+_{a,n,1}$. Then, we have

$$\begin{aligned}\frac{\partial^2 S}{\partial s^2}(s,t) &= \frac{(3s-t) - 2(t-s)^3}{\left((t-s)^2+1\right)^{3/2}} \\ &= \frac{(s+t) - 2(t-s)\left((t-s)^2+1\right)}{\left((t-s)^2+1\right)^{3/2}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 S}{\partial t^2}(s,t) &= \frac{(3t-s) + 2(t-s)^3}{\left((t-s)^2+1\right)^{3/2}} \\ &= \frac{(s+t) + 2(t-s)\left((t-s)^2+1\right)}{\left((t-s)^2+1\right)^{3/2}},\end{aligned}$$

and

$$\frac{\partial^2 S}{\partial s \partial t}(s,t) = \frac{\partial^2 S}{\partial t \partial s}(s,t) = -\frac{s+t}{\left((t-s)^2+1\right)^{3/2}}.$$

By using a theorem in the hypergeometric function theory, we see

$$g_k(x) = (x/2)\times\left(\left(\sqrt{1+(\frac{1}{2x})^2} + \frac{1}{2x}\right)^{2k} + \left(\sqrt{1+(\frac{1}{2x})^2} - \frac{1}{2x}\right)^{2k}\right).$$

If we put

$$A = A(x) := \sqrt{1+(\frac{1}{2x})^2} + \frac{1}{2x},$$

then

$$\sqrt{1+(\frac{1}{2x})^2} - \frac{1}{2x} = \frac{1}{A}$$

and

$$x = \frac{1}{A - \frac{1}{A}}.$$

Now, we put for $i \in \mathbb{Z}$,

$$\alpha_i = \alpha_i(x) := A^i + \frac{1}{A^i}, \quad \beta_i = \beta_i(x) := A^i - \frac{1}{A^i}.$$

Then, we easily check that

$$\alpha_i = \alpha_{-i}, \quad \beta_i = -\beta_{-i}, \quad \alpha_0 = 2, \quad \beta_0 = 0,$$

$$\alpha_i\alpha_j = \alpha_{i+j} + \alpha_{i-j}, \quad \beta_i\beta_j = \alpha_{i+j} - \alpha_{i-j},$$

$$\beta_i^2 + 4 = \alpha_i^2, \quad \alpha_i\beta_i = \beta_{2i},$$

and

$$g_k(x) = \frac{\alpha_{2k}}{2\beta_1}.$$

Moreover we have

$$\begin{aligned}(g_k(x) - g_{k-1}(x))^2 + 1 &= \left(\frac{\alpha_{2k} - \alpha_{2k-2}}{2\beta_1}\right)^2 + 1 \\ &= \left(\frac{\beta_{2k-1}\beta_1}{2\beta_1}\right)^2 + 1 \\ &= \left(\frac{\alpha_{2k-1}}{2}\right)^2.\end{aligned}$$

From the above, we can write simply

$$\frac{\partial^2 S}{\partial s^2}\left(g_{k-1}(x), g_k(x)\right) = \frac{2(2\alpha_1 - \alpha_{4k-1} + \alpha_{4k-3})}{\beta_1\alpha_{2k-1}^2},$$

$$\frac{\partial^2 S}{\partial t^2}\left(g_{k-1}(x), g_k(x)\right) = \frac{2(2\alpha_1 + \alpha_{4k-1} - \alpha_{4k-3})}{\beta_1\alpha_{2k-1}^2},$$

and

$$\frac{\partial^2 S}{\partial s \partial t}\left(g_{k-1}(x), g_k(x)\right) = \frac{-4\alpha_1}{\beta_1\alpha_{2k-1}^2}.$$

Next, we consider $g_k'(x)$. Since

$$\begin{aligned}A' &= \frac{\frac{1}{2x}(-\frac{1}{2x^2})}{\sqrt{1+(\frac{1}{2x})^2}} - \frac{1}{2x^2} \\ &= -\frac{\sqrt{1+(\frac{1}{2x})^2} + \frac{1}{2x}}{2x^2\sqrt{1+(\frac{1}{2x})^2}} \\ &= -\frac{\beta_1^2 A}{A + \frac{1}{A}} \\ &= -\frac{\beta_1^2 A}{\alpha_1},\end{aligned}$$

we see that for $i \in \mathbb{N}$,

$$\begin{aligned}\alpha_i' &= A'(iA^{i-1} - iA^{-i-1}) \\ &= -\frac{i\beta_1^2}{\alpha_1}\left(A^i - \frac{1}{A_i}\right) \\ &= -\frac{i\beta_1^2\beta_i}{\alpha_1}.\end{aligned}$$

Thus, together with $1/\beta_1 = x$, we see that

$$\begin{aligned}g_k'(x) &= \frac{1}{2}\left(\alpha_{2k} + \frac{\alpha_{2k}'}{\beta_1}\right) \\ &= \frac{\alpha_1\alpha_{2k} - 2k\beta_1\beta_{2k}}{2\alpha_1} \\ &= \frac{(1-2k)\alpha_{2k+1} + (1+2k)\alpha_{2k-1}}{2\alpha_1}.\end{aligned}$$

Now we consider, for $n \in \mathbb{N}$ and $x > 0$, the Hessian matrix $H_n(x)$ of

$$\tilde{S}_{g_n(x),n,1}(x_0, x_1, x_2, \ldots, x_{n-1})$$

at $(x, g_1(x), g_2(x), \ldots, g_{n-1}(x))$.

Lemma 2. *We have*

$$\det H_n(x) = 4^n \left(\frac{\alpha_1}{\beta_1}\right)^n \frac{g_n'(x)}{\alpha_1^2\alpha_2^2\cdots\alpha_{2n-1}^2}.$$

Proof. We prove this by induction. In the cases where $n = 1, 2$, we obtain the lemma by direct calculation. We assume that the lemma holds for $1, 2, \ldots, n-1$, here $n \geq 3$.

Recalling that

$$\begin{aligned}&\tilde{S}_{g_n(x),n,1}(x_0, x_1, x_2, \ldots, x_{n-1})\\ &= S(x_0, x_1) + S(x_1, x_2) + \cdots\\ &\quad + S(x_{n-2}, x_{n-1}) + S(x_{n-1}, g_n(x)),\end{aligned}$$

$H_n(x) = (h_{i,j})_{i,j=1,2,\ldots,n}$ is expressed as

$$\begin{aligned}h_{1,1} &= \frac{\partial^2 S}{\partial s^2}(g_0(x), g_1(x)),\\ h_{i,i} &= \frac{\partial^2 S}{\partial t^2}(g_{i-2}(x), g_{i-1}(x)) + \frac{\partial^2 S}{\partial s^2}(g_{i-1}(x), g_i(x))\end{aligned}$$

for $i = 2, 3, \ldots, n$,

$$h_{i,i+1} = h_{i+1,i} = \frac{\partial^2 S}{\partial s\partial t}(g_{i-1}(x), g_i(x))$$

for $i = 1, 2, \ldots, n$, and

$$h_{i,j} = 0$$

if $|i - j| \geq 2$. Consequently, we have

$$\begin{aligned}&\det H_n(x)\\ &= \det H_{n-1}(x)\\ &\times \left(\frac{\partial^2 S}{\partial t^2}(g_{n-2}(x), g_{n-1}(x)) + \frac{\partial^2 S}{\partial s^2}(g_{n-1}(x), g_n(x))\right)\\ &- \det H_{n-2}(x) \times \left(\frac{\partial^2 S}{\partial s\partial t}(g_{n-2}(x), g_{n-1}(x))\right)^2.\end{aligned}$$

Omitting the middle formulas, we see

$$\begin{aligned}&\frac{\partial^2 S}{\partial t^2}(g_{n-2}(x), g_{n-1}(x)) + \frac{\partial^2 S}{\partial s^2}(g_{n-1}(x), g_n(x))\\ &= \frac{2(2\alpha_1 + \alpha_{4n-5} + \alpha_{4n-7})}{\beta_1\alpha_{2n-3}^2} + \frac{2(2\alpha_1 - \alpha_{4n-1} + \alpha_{4n-3})}{\beta_1\alpha_{2n-1}^2}\\ &= \frac{4(2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1)}{\beta_1\alpha_{2n-3}^2\alpha_{2n-1}^2},\end{aligned}$$

and from the induction hypothesis,

$$\begin{aligned}&\det H_{n-1}(x)\\ &\times \left(\frac{\partial^2 S}{\partial t^2}(g_{n-2}(x), g_{n-1}(x)) + \frac{\partial^2 S}{\partial s^2}(g_{n-1}(x), g_n(x))\right)\\ &= \left(\frac{\alpha_1}{\beta_1}\right)^{n-1} \frac{4^{n-1}\times 2}{\alpha_1^2\alpha_3^2\cdots\alpha_{2n-5}^2\alpha_{2n-3}^4\alpha_{2n-1}^2\alpha_1}\\ &\times ((3-2n)\alpha_{2n-1} + (2n-1)\alpha_{2n-3})\\ &\times (2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1).\end{aligned}$$

Moreover,

$$\begin{aligned}&((3-2n)\alpha_{2n-1} + (2n-1)\alpha_{2n-3})\\ &\times (2\alpha_{4n-3} + 2\alpha_{4n-5} - \alpha_5 + \alpha_3 + 4\alpha_1)\\ &= 2(3-2n)\alpha_{6n-4} + 4\alpha_{6n-6} + 2(2n-1)\alpha_{6n-8}\\ &\quad + (2n-3)\alpha_{2n+4} - 4(n-1)\alpha_{2n+2} + (9-2n)\alpha_{2n}\\ &\quad + 12\alpha_{2n-2} + (2n+5)\alpha_{2n-4} + 4(n-1)\alpha_{2n-6}\\ &\quad - (2n-1)\alpha_{2n-8}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\det H_{n-2}(x) \times \left(\frac{\partial^2 S}{\partial s\partial t}(g_{n-2}(x), g_{n-1}(x))\right)^2\\ &= \left(\frac{\alpha_1}{\beta_1}\right)^{n-1} \frac{4^{n-1}\times 2}{\alpha_1^2\alpha_3^2\cdots\alpha_{2n-5}^2\alpha_{2n-3}^4\alpha_1}\\ &\times ((5-2n)\alpha_{2n-2} + 2\alpha_{2n-4} + (2n-3)\alpha_{2n-6}),\end{aligned}$$

and thus

$$\begin{aligned}&\det H_n(x)\\ &= \left(\frac{\alpha_1}{\beta_1}\right)^{n-1} \frac{4^{n-1}\times 2}{\alpha_1^2\alpha_3^2\cdots\alpha_{2n-5}^2\alpha_{2n-3}^4\alpha_{2n-1}^2\alpha_1}\\ &\times \{(-2n+1)\alpha_{6n-4} + 2\alpha_{6n-6} + (2n+1)\alpha_{6n-8}\\ &+ (-4n+2)\alpha_{2n+2} + 4\alpha_{2n} + (4n+2)\alpha_{2n-2}\\ &+ (2n+1)\alpha_{2n-4} + 2\alpha_{2n-6} + (-2n+1)\alpha_{2n-8}\}.\end{aligned}$$

On the other hand,

$$g_n'(x) = \frac{(1-2n)\alpha_{2n+1} + (2n+1)\alpha_{2n-1}}{2\alpha_1},$$

and by direct calculation, we obtain that

$$\begin{aligned}&\alpha_1\alpha_{2n-3}^2((1-2n)\alpha_{2n+1} + (2n+1)\alpha_{2n-1})\\ &= (-2n+1)\alpha_{6n-4} + 2\alpha_{6n-6} + (2n+1)\alpha_{6n-8}\\ &\quad + (-4n+2)\alpha_{2n+2} + 4\alpha_{2n} + (4n+2)\alpha_{2n-2}\\ &\quad + (2n+1)\alpha_{2n-4} + 2\alpha_{2n-6} + (-2n+1)\alpha_{2n-8}.\end{aligned}$$

This completes the proof. □

6. Proof of Theorem 2

The following lemma is well-known.

Lemma 3. *A symmetric $n \times n$ matrix $A = (a_{ij})_{i,j=1,\ldots n}$ is positive definite if and only if $\det A_k > 0$ for any $k = 1, 2, \ldots, n$, where $A_k := (a_{ij})_{i,j=1,2,\ldots,k}$.*

Proof of Theorem 2. Lemma 2 implies that if $x > \xi_{n,1}$, then $\det H_n(x) > 0$. Moreover, as is seen in Remark 2, $\xi_{n,1} > \xi_{n-1,1} > \cdots > \xi_{1,1}$ and thus if $x > \xi_{n,1}$, then $\det H_k(x) > 0$ for $k = 1, 2, \ldots, n$. Together with Lemma 3, we see that $H_n(x_{a,n,1}^+)$ is positive definite and

$$2D_{n,1}(x_{a,n,1}^+), g_{1,\ell}(x_{a,n,1}^+), \ldots, g_{n-1,\ell}(x_{a,n,1}^+), a)$$

is stable for $a > \eta_{n,1}$. □

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References

[1] Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series **55**, 1964.

[2] U. Dierkes, S. Hildebrandt and F. Sauvigny: Minimal Surfaces, A Series of Comprehensive Studies in Mathematics **339**, Springer-Verlag, 2010.

[3] J. A. Thorpe: Elementary Topics in Differential Geometry, Undergraduate Texts in Mathematics, Springer-Verlag, 1979.

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