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Asymptotic behavior of blow-up solutions to a degenerate parabolic equation

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Abstract. We consider the Dirichlet problems for a degenerate parabolic equation, $u_t = u^\delta(\Delta u + \lambda u)$ in a bounded domain in $\mathbb{R}^n$ with a smooth boundary. It has been known that if $\delta \geq 2$ then there exists u which blows up faster than the rate of $(T-t)^{-1/\delta}$, where T is the blow-up time of u. The solutions are called "Type 2". In this paper we investigate features for asymptotic behavior of "Type 2" solutions for the case of $\delta = 2$ and $\delta > 2$.

Keywords. degenerate parabolic equations, blow-up, asymptotic behavior, type 2, eventual monotonicity

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Suppose that a function $u_0 \in C^\infty(\Omega) \bigcap C(\overline{\Omega})$ satisfies

$$u_0 > 0 \quad \text{in } \Omega \quad \text{and} \quad u_0(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{1.1}$$

Then, we consider the following initial-boundary value problems.

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^\delta(\Delta u + \lambda u) & x \in \Omega, t > 0, \\ u(x,0) = u_0(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.2}$$

where $\delta > 0$ and λ is greater than the first eigenvalue, $\lambda_1(\Omega)$ of $-\Delta$ in Ω, that is,

$$\lambda > \lambda_1(\Omega).$$

In this paper we discuss classical solutions which are approximated by functions u_ε solving the following problems:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = u_\varepsilon^\delta(\Delta u_\varepsilon + \lambda u_\varepsilon) & x \in \Omega, t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon & x \in \overline{\Omega}, \\ u_\varepsilon(x,t) = \varepsilon & x \in \partial\Omega, t \geq 0. \end{cases} \tag{1.3}$$

It has been proved that each of them blows up at a finite time, that is, for any solutions approximated by u_ε, there exists $T > 0$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} u(x,t)$. (For instance, see [2].)

Here the constant T is called "the blow-up time" of u. Moreover it has been known that they are classified into two types by their blow-up rates as follows.

Definition 1. Let u and T be a solution of (1.2) and the blow-up time of u, respectively. Then u is called "Type 1" if it satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{\delta}} \|u(\cdot,t)\|_\infty < +\infty \tag{1.4}$$

and u is called "Type 2" if it satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{\delta}} \|u(\cdot,t)\|_\infty = +\infty. \tag{1.5}$$

Precisely, It has been proved that if $\delta \geq 2$ then there exists solutions of (1.2) which are "Type 2" and if $0 < \delta < 2$ then u is "Type 1". (See [1], [3], [4], [6], [9], [10] and so force.) Our purpose of this paper is to investigate asymptotic behavior of "Type 2" solutions for $\delta \geq 2$.

First, we consider the case of $\delta = 2$. S. Angenent gave important asymptotic profiles in [3] for curve shortening problems as follows.

$$v_t = v^2(v_{xx} + v) \quad \text{with periodic boundary conditions.}$$

The first profile is that there exists $t_0 > 0$ such that $v_t(x,t) = v(x,t)^2\Big(v_{xx}(x,t) + v(x,t)\Big) > 0$ for any x and $t > t_0$. This property is called "eventual monotonicity". Secondly, it was proved that $V(x,t) := \dfrac{v(x,t)}{\|v(\cdot,t)\|_\infty}$ satisfies

$$V(x,t) \to \cos(x - x_0) \quad \text{if } |x - x_0| \leq \frac{\pi}{2}, \tag{1.6}$$

as $t \nearrow T$ for some x_0. Besides, an upper bound for blow-up rates of solutions was given. In order to prove its existence, Angenent investigated relations between $\displaystyle\int_\Omega \log v(x,t)\,dx$ and $\log(T-t)$.

Their results hold under a special situation in curve shortening problems but we have not known whether they can be proved without any special assumptions.

In Section 3 we will discuss asymptotic behavior of solution to (1.2) without any special assumptions in the case of $\delta = 2$. Our method is based on behavior of $\int_\Omega u^{1-\delta} u_t \, dx$ as t tends to the blow-up time of u. Precisely, we will prove in Section 3 that if $\delta = 2$ then $2(T-t)\int_\Omega u^{-1} u_t \, dx$ becomes positive and has an upper bound after a finite time. In particular, the positivity implies that there exists $t_0 \in (0, T)$ such that $\int_\Omega u^{-1} u_t \, dx = \int_\Omega u(\Delta u + \lambda u) \, dx > 0$ for $t \in [t_0, T)$ and we call this property "weak eventual monotonicity".

The positivity and the upper boundedness of $2(T - t)\int_\Omega u^{-1} u_t \, dx$ on $\delta = 2$ lead to two profiles for asymptotic behavior of solutions to (1.2). One is that there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ and

$$\frac{\int_\Omega |\nabla U(x, t_k)|^2 \, dx}{\int_\Omega U(x, t_k)^2 \, dx} \to \lambda \quad \text{as } k \to \infty, \tag{1.7}$$

where $U(x, t) := \dfrac{u(x, t)}{\|u(\cdot, t)\|_\infty}$. We can regard (1.7) as a generalization of (1.6). Another is that if $\delta = 2$ and $\log u_0 \in L^1(\Omega)$ then $\int_\Omega \log\left[(T - t)^{\frac{1}{2}} u(x, t)\right] dx$ has an upper bound, that is, there exists a positive constant $C > 0$ such that

$$\int_\Omega \log\left[(T - t)^{\frac{1}{2}} u(x, t)\right] dx \le C \tag{1.8}$$

for $t \in [t_0, T)$, where C depends on T and u_0 only. It may give a important profile for blow-up rates of "Type 2" that (1.5) holds for $\delta = 2$ but satisfies (1.8).

In Section 4 the case of $\delta > 2$ will be discussed. We expect that behavior of solutions for $\delta > 2$ may be a little different from $\delta = 2$. In the situation, we investigate features of $\int_\Omega u^{1-\delta} u_t \, dx$, $\int_\Omega u^{2-\delta} \, dx$ and others by numerical computing. And then we will present two conjectures for asymptotic behavior of solutions of (1.2) by numerical observations. Moreover, we will prove under their conjectures that $\delta(T-t)^{\frac{\delta}{2}} \int_\Omega u^{1-\delta} u_t \, dx$ has an upper bound after a finite time and (1.7) holds for some sequences.

2. Some Lemmas

Let u be a solution of (1.2) approximated by u_ε solving (1.3). The following two lemmas have been well-known in previous studies for asymptotic behavior of u.

Lemma 1. *It holds that*

$$u_t(x, t) \ge -\frac{1}{\delta t} u(x, t) \quad \textit{for } x \in \Omega \textit{ and } t \in (0, T).$$

Lemma 2. *If $\{t_k\}_{k\in\mathbb{N}}$ is a sequence such that $t \nearrow T$ as $k \to \infty$, then there is $\mu \in (0, 1)$ and a set $S_\mu \subset \Omega$ with positive Lebesgue measure such that*

$$u(x, t) \ge \mu \|u(\cdot, t)\|_\infty \quad \textit{for } x \in S_\mu \textit{ and } t \in (0, T)$$

for all k from an appropriate subsequence. In particular, the Lebesgue measure of $\left\{x \in \Omega \middle| \limsup_{t\nearrow T} u(x, t) = \infty\right\}$ is positive.

They are very important properties for our purpose and can be proved by the maximum principle for parabolic equations. (For instance, see [9] and [10] for their proofs.)

Moreover, we give another important property for u.

Lemma 3. *If $u_0 \in C^\infty(\Omega) \bigcap C(\overline{\Omega})$ satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$ then*

$$\left(\int_\Omega u^{1-\delta} u_t \, dx\right)^2 < \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta} (u_t)^2 \, dx \tag{2.1}$$

for any $t \in (0, T)$.

Remark 1. In Lemma 3, $u^{1-\delta} u_t$, $u^{-\delta}(u_t)^2$ and $u^{2-\delta}$ is integrable in Ω for any $t \in (0, T)$ because u is a classical solution of $u_t = u^\delta(\Delta u + \lambda u)$.

Remark 2. When $\delta = 2$, the assumption $u_0^{2-\delta} \in L^1(\Omega)$ is not required and then we can prove

$$\left(\int_\Omega u^{-1} u_t \, dx\right)^2 < |\Omega| \int_\Omega u^{-2} (u_t)^2 \, dx,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Proof. By Schwartz's inequality,

$$\left(\int_\Omega u^{1-\delta} u_t \, dx\right)^2 \le \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta} (u_t)^2 \, dx. \tag{2.2}$$

Note that $\int_\Omega u^{2-\delta} \, dx = |\Omega|$ if $\delta = 2$. Now, $u^{2-\delta} \not\equiv u^{-\delta}(u_t)^2$ because $u^{-1} u_t$ is not constant in Ω. Hence, the equality in (2.2) does not hold and we get this lemma. □

Remark 3. (2.1) implies that When $\delta > 2$, if u_0 satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$ then

$$E(t) := \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta} (u_t)^2 \, dx - \left(\int_\Omega u^{1-\delta} u_t \, dx\right)^2 > 0. \tag{2.3}$$

We will discuss the ratio of $E(t)$ to $\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2$ near the blow-up time in Section 4.

Let $\{u_{0,n}\}_{n\in\mathbb{N}} \in C^\infty(\Omega)$ be such that $\|u_{0,n}-u_0\|_{L^\infty(\Omega)} < \frac{1}{k}$ and $\alpha_n\Theta \le u_{0,n} \le \beta_n\Theta$ holds with constants $0<\alpha_n<\beta_n$, where Θ denotes the principal eigenfunction of $-\Delta$ in Ω with $\max_\Omega \Theta = 1$. Then it has been shown in [9] that if $\delta>1$ then the problems

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = u_n^\delta(\Delta u_n + \lambda u_n) & x\in\Omega, t\in(0,T(u_n)),\\ u_n(x,0)=u_{0,n}(x) & x\in\overline{\Omega},\\ u_n(x,t)=0 & x\in\partial\Omega, t\ge 0,\end{cases}$$

have unique solutions $u_n \in C(\overline{\Omega}\times[0,T(u_n))\bigcap C^\infty(\Omega\times(0,T(u_n)))$, $T(u_n)\to T$, which may be approximated by function $u_{n,\varepsilon}$ solving (1.3) with $u_{n,\varepsilon} = u_n+\varepsilon$ on the parabolic boundary. As $n\to\infty$, $u_n\to u$ in the topology of $C_{loc}(\overline{\Omega}\times[0,T))\bigcap C^\infty_{loc}(\Omega\times(0,T))$.

Furthermore, u_n have the property that for each $n\in\mathbb{N}$ and each $\tau>0$ there is a constant $c(n,\tau)>0$ such that

$$|\mathrm{dist}\,(x,\Omega)^{\alpha-1+j(2-\delta)}\partial_x^\alpha\partial_t^j u_n(x,t)| \le c(n,\tau)$$

for $j=0,1$, $\alpha=0,1,2$ and $(x,t)\in\Omega\times(0,T-\tau)$. In particular, $|\nabla(u_n)_t| \le \mathrm{dist}\,(x,\partial\Omega)^{\delta-2}c(n,\tau)$ for $(x,t)\in\Omega\times(0,T-\tau)$ and u_n satisfies

$$\begin{aligned}\int_\Omega u_n^{-\delta}\Big((u_n)_t\Big)^2dx &= \int_\Omega(\Delta u_n+\lambda u_n)(u_n)_t\,dx\\ &= \int_\Omega\Big(-\nabla u_n\cdot\nabla(u_n)_t + \lambda u_n(u_n)_t\Big)dx\\ &= \frac12\left(\int_\Omega(-|\nabla u_n|^2+\lambda u_n^2)\,dx\right)_t\\ &= \frac12\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)_t. \end{aligned} \tag{2.4}$$

Therefore, by Remark 2, we obtain the following lemma.

Lemma 4. *If $\delta=2$ then*

$$2\left(\int_\Omega u_n^{-1}(u_n)_t\,dx\right)^2 < |\Omega|\left(\int_\Omega u_n^{-1}(u_n)_t\,dx\right)_t$$

for any $t\in(0,T(u_n))$.

And Lemma 4 leads to the following corollary.

Corollary 1. *If $\delta=2$ then $\int_\Omega u_n^{-1}(u_n)_t\,dx$ is strictly increasing with respect to t.*

Proof. By Lemma 4, if $\delta=2$ then

$$\left(\int_\Omega u_n^{-1}(u_n)_t\,dx\right)_t > \frac{2}{|\Omega|}\left(\int_\Omega u_n^{-1}(u_n)_t\,dx\right)^2 > 0$$

for any $t\in(0,T(u_n))$. □

3. The case of $\delta = 2$

3.1. The Main Theorem for $\delta = 2$

We consider the case of $\delta=2$, that is, the following problem.

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^2(\Delta u+\lambda u) & (x,t)\in\Omega\times(0,T),\\ u(x,0)=u_0(x) & x\in\overline{\Omega},\\ u(x,t)=0 & (x,t)\in\partial\Omega\times(0,T),\end{cases} \tag{3.1}$$

where T is the blow-up time of u. Then we can prove the following theorem.

Theorem 1. *Let $\delta=2$ and u be a solution approximated by u_ε solving (1.3). Suppose that $u_0\in C^\infty(\Omega)\bigcap C(\overline{\Omega})$ satisfies (1.1) and $\log u_0 \in L^1(\Omega)$. Then there exists $t_0>0$ such that*

$$0 < 2(T-t)\int_\Omega u^{-1}u_t\,dx \le |\Omega|$$

for any $t\in[t_0,T)$, where $T>0$ is the blow-up time of u.

Remark 4. The left side inequality in Theorem 1 implies that $\int_\Omega u^{-1}u_t\,dx = \int_\Omega u(\Delta u+\lambda u)\,dx > 0$ for $t\in[t_0,T)$. In this paper we call this property "weak eventual monotonicity".

Proof of Theorem 1. Let $\tau>0$ be small enough and $\{t_k\}_{k=1,2,\cdots}$ be a sequence with $t_k>\tau$, $t_k\nearrow T$ and $\|u(\cdot,t_k)\|_\infty \nearrow \infty$ as $k\to\infty$. Since Lemma 1 implies $(\log u)_t \ge -\frac{1}{2t}$, it holds that

$$\log u(x,t_k) - \log u(x,\tau) \ge -\frac12(\log t_k - \log\tau) > -\frac12\log\frac{T}{\tau}.$$

By Lemma 2, for an appropriate subsequence, there exists $\mu\in(0,1)$ and a set S_μ such that $|S_\mu|>0$ and

$$\begin{aligned}&\int_\Omega \log u(x,t_k)\,dx\\ &\quad= \int_{S_\mu}\log u(x,t_k)\,dx + \int_{\Omega\setminus S_\mu}\log u(x,t_k)\,dx\\ &\quad\ge |S_\mu|\log(\mu\|u(\cdot,t_k)\|_\infty)\\ &\qquad - \frac12|\Omega\setminus S_\mu|\log\frac{T}{\tau} + \int_{\Omega\setminus S_\mu}\log u(x,\tau)\,dx.\end{aligned}$$

Since the last term is bounded for a fixed positive number $\tau\in(0,T)$ because of $\log u_0\in L^1(\Omega)$, we can get

$$\int_\Omega \log u(x,t_k)\,dx \to \infty \quad \text{as } k\to\infty \text{ for any } n\in\mathbb{N}$$

which implies that there exists t_0 such that

$$\int_\Omega u(x,t_0)^{-1}u_t(x,t_0)\,dx = \frac{d}{dt}\int_\Omega \log u(x,t_0)\,dx > 0.$$

Hence, by Corollary 1,

$$\int_\Omega u^{-1}u_t\,dx > 0 \quad \text{for any } t \in [t_0, T).$$

On the other hands, when $\delta = 2$, Lemma 4 leads to

$$\frac{2}{|\Omega|} + \left[\left(\int_\Omega u_n^{-1}(u_n)_t\,dx\right)^{-1}\right]_t < 0 \quad \text{for } t \in (0, T(u_n))$$

for u_n defined in Section 2. Integrating from t to s, we have

$$\frac{2(s-t)}{|\Omega|} + \left(\int_\Omega u_n(x,s)^{-1}(u_n)_t(x,s)\,dx\right)^{-1} - \left(\int_\Omega u_n(x,t)^{-1}(u_n)_t(x,t)\,dx\right)^{-1} < 0.$$

Since

$$\int_\Omega u_n(x,s)^{-1}(u_n)_t(x,s)\,dx > \int_\Omega u_n(x,t)^{-1}(u_n)_t(x,t)\,dx > 0$$

if $t_0 \le t < s < T(u_n)$, it holds that

$$2(s-t)\int_\Omega u_n(x,t)^{-1}(u_n)_t(x,t)\,dx < |\Omega|,$$

where $t_0 \le t < s < T(u_n)$. Then it is verified by letting $s \nearrow T(u_n)$ that

$$2(T(u_n)-t)\int_\Omega u_n(\Delta u_n + \lambda u_n)\,dx = 2(T(u_n)-t)\int_\Omega u_n^{-1}(u_n)_t\,dx \le |\Omega|$$

for any $t \in [t_0, T(u_n))$.

Now, we mentioned in Section 2 that $T(u_n) \to T$ and $u_n \to u$ in the topology of $C^0_{loc}(\overline{\Omega}\times[0,T)) \bigcap C^\infty_{loc}(\Omega\times(0,T))$ as $n \to \infty$. This implies that $u_n(\cdot,t) \to u(\cdot,t)$ uniformly in $\overline{\Omega}$ and $\Delta u_n(\cdot,t) \rightharpoonup \Delta u(\cdot,t)$ weakly in $C^*(\Omega)$ as $n \to \infty$ for any $t \in [t_0, T)$. Then we have

$$2(T-t)\int_\Omega u^{-1}u_t\,dx = 2(T-t)\int_\Omega u(\Delta u + \lambda u)\,dx \le |\Omega|$$

which completes this proof. □

3.2. Behavior near the Blow-up Time

We discuss some important profiles for asymptotic behavior of $U(x,t) := \dfrac{u(x,t)}{\|u(\cdot,t)\|_\infty}$, where u is a "Type 2" blow-up solution to (3.1).

First, suppose that u_0 satisfies (1.1) and $\Delta u_0 + \lambda u_0 \ge 0$ in Ω. Then it is easily verified by the maximum principle that $u > 0$, $\Delta u + \lambda u \ge 0$ and $U(\Delta U + \lambda U) \ge 0$ in $\Omega \times (0,T)$. Besides, Theorem 1 leads to

$$\begin{aligned} 0 < \int_\Omega U(\Delta U + \lambda U)\,dx &= \frac{1}{\|u(\cdot,t)\|_\infty^2}\int_\Omega u(\Delta u + \lambda u)\,dx \\ &= \frac{1}{\|u(\cdot,t)\|_\infty^2}\int_\Omega u^{-1}u_t\,dx \\ &\le \frac{|\Omega|}{2(T-t)\|u(\cdot,t)\|_\infty^2} \end{aligned}$$

for $t \in [t_0, T)$. In addition, "Type 2" means

$$\limsup_{t \nearrow T}(T-t)^{\frac{1}{2}}u(x,t) = +\infty.$$

Therefore, if u is "Type 2" and u_0 satisfies $\Delta u_0 + \lambda u_0 \ge 0$ in Ω, then we obtain that there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and U satisfies

$$\Delta U(x,t_k) + \lambda U(x,t_k) \to 0 \text{ or } U(x,t_k) \to 0$$

for almost all of $x \in \Omega$ as $k \to \infty$.

On the other hands, we assume (1.1) and $\log u_0 \in L^1(\Omega)$ only. Then Theorem 1 can lead to the following corollary such that we can regard as a generalization of the profile for asymptotic behavior of $U(x,t)$ near the blow-up time.

Corollary 2. *Let $\delta = 2$ and u be a "Type 2" solution of (3.1) approximated by u_ε solving (1.3). Suppose that $u_0 \in C^\infty(\Omega) \bigcap C(\overline{\Omega})$ satisfies (1.1) and $\log u_0 \in L^1(\Omega)$. Then there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and $U(x,t) := \dfrac{u(x,t)}{\|u(\cdot,t)\|_\infty}$ satisfies*

$$\frac{\int_\Omega |\nabla U(x,t_k)|^2\,dx}{\int_\Omega U(x,t_k)^2\,dx} \to \lambda \quad \text{as } k \to \infty.$$

Proof. If $\delta = 2$, then

$$\int_\Omega u^{-1}u_t\,dx = \int_\Omega u(\Delta u + \lambda u)\,dx = \int_\Omega \Big(-|\nabla u|^2 + \lambda u^2\Big)\,dx.$$

By Theorem 1, we have

$$0 < \int_\Omega \Big(-|\nabla u(x,t)|^2 + \lambda u(x,t)^2\Big)\,dx \le \frac{|\Omega|}{2(T-t)}$$

for $t \in [t_0, T)$. Hence, $U(x,t) = \dfrac{u(x,t)}{\|u(\cdot,t)\|_\infty}$ satisfies

$$\int_\Omega |\nabla U(x,t)|^2\,dx < \lambda\int_\Omega U(x,t)^2\,dx \tag{3.2}$$

and

$$\lambda\int_\Omega U(x,t)^2\,dx - \frac{|\Omega|}{2(T-t)\|u(\cdot,t)\|_\infty^2} \le \int_\Omega |\nabla U(x,t)|^2\,dx. \tag{3.3}$$

Since if $\delta = 2$ and u is Type 2, then

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{2}} \|u(\cdot,t)\|_\infty = \infty,$$

there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that

$$t_k \nearrow T \text{ and } (T-t_k)\|u(\cdot,t_k)\|_\infty^2 \to \infty \quad \text{as } k \to \infty.$$

Furthermore, Lemma 2 implies that

$$0 < \mu^2 |S_\mu| \le \int_\Omega U(x,t_k)^2\,dx \le |\Omega| \quad \text{for any } k. \tag{3.4}$$

Therefore, we have

$$\frac{\displaystyle\int_\Omega |\nabla U(x,t_k)|^2\,dx}{\displaystyle\int_\Omega U(x,t_k)^2\,dx} \to \lambda \quad \text{as } k \to \infty. \qquad \square$$

3.3. A Profile for blow-up rates

We mentioned in Section 1 that there exists "Type 2" solutions of (3.1) for $\delta = 2$ which satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{2}} \|u(\cdot,t)\|_\infty = \infty \tag{3.5}$$

The following corollary gives another feature for asymptotic behavior of $(T-t)^{\frac{1}{2}} u(x,t)$ as $t \nearrow T$.

Corollary 3. *Let $\delta = 2$ and u be a solution of* (3.1) *approximated by u_ε solving* (1.3). *Suppose that $u_0 \in C^\infty(\Omega) \bigcap C(\overline{\Omega})$ satisfies* (1.1) *and $\log u_0 \in L^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\int_\Omega \log\left[(T-t)^{\frac{1}{2}} u(x,t)\right] dx \le C$$

for $t \in [t_0, T)$, where C depends on T and u_0 only.

Proof. Since Theorem 1 implies

$$0 < \frac{d}{dt}\int_\Omega \log u\,dx \le \frac{|\Omega|}{2(T-t)} \quad \text{for any } t \in [t_0, T),$$

we have

$$\frac{d}{dt}\int_\Omega \log\left[(T-t)^{\frac{1}{2}} u(x,t)\right] dx \le 0 \quad \text{for any } t \in [t_0, T).$$

Therefore,

$$\int_\Omega \log\left[(T-t)^{\frac{1}{2}} u(x,t)\right] dx \le \int_\Omega \log\left[T^{\frac{1}{2}} u_0(x)\right] dx$$

which completes this proof. $\square$

4. The case of $\delta > 2$

4.1. Conjectures for $\delta > 2$

For the case of $\delta > 2$, we expect that behavior of blow-up solutions may be a little different from $\delta = 2$. In this section we try to establish some profiles for $\delta > 2$ by numerical computing.

Precisely, we investigated features for asymptotic behavior of blow-up solutions to (1.2) by numerical computing with many kinds of initial data. Note that all of our computations are based on the scheme provided in [5].

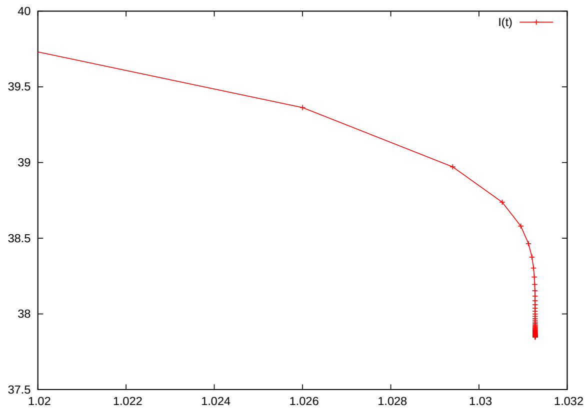
For example, we consider the case of $\lambda = 1.0$, $\Omega = (0, 20) \in \mathbb{R}^1$ and

$$u_0(x) = \frac{1}{100}\left(\sin\frac{\pi}{20}x\right)\left(\sin^2\frac{3\pi}{20}x\right). \tag{4.1}$$

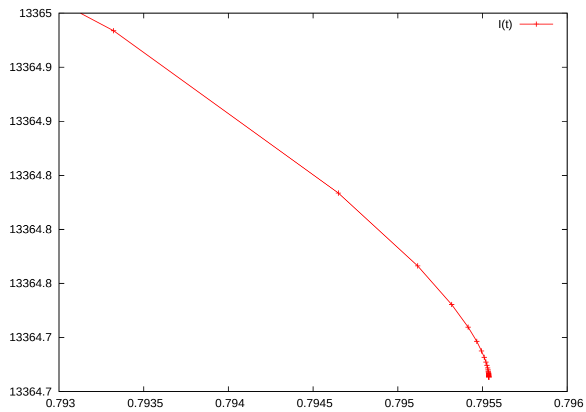
Figure 1 (a) and (b) are behavior of

$$I(t) := \int_\Omega u^{2-\delta}\,dx$$

for $\delta = 2.5$ and $\delta = 4.0$ near the blow-up time, respectively.



(a) Behavior of $I(t)$ for $\delta = 2.5$



(b) Behavior of $I(t)$ for $\delta = 4.0$

Figure 1: Examples of behavior of $I(t) = \displaystyle\int_\Omega u^{2-\delta}\,dx$

They show that $I(t)$ is decreasing near the blow-up time. Since $\frac{d}{dt}I(t) = (2-\delta)\int_\Omega u^{1-\delta}u_t\,dx$, this observation implies that if $\delta > 2$ then $\int_\Omega u^{1-\delta}u_t\,dx$ should be positive near the blow-up time, that is, u should satisfy the same property as "weak eventual monotonicity" in Remark 4 for the case of $\delta = 2$. We are computing numerically with so many initial data but we have not gotten what does not satisfy "weak eventual monotonicity".

Remark 5. The property of "weak eventual monotonicity" means that $\int_\Omega u^{1-\delta}u_t\,dx$ should change positive even if it is negative near the initial time $t = 0$, that is, $I(t) = \int_\Omega u^{2-\delta}\,dx$ should be strictly decreasing after a finite time. For instance, Figure 2 is an example of behavior near the initial time of $I(t) = \int_\Omega u^{2-\delta}\,dx$ with the initial data $u_0(x) = K^2 - x^2$, where $\Omega = (-K, K) \in \mathbb{R}^1$, $\delta = 2.5$ and $K = 1.575$.

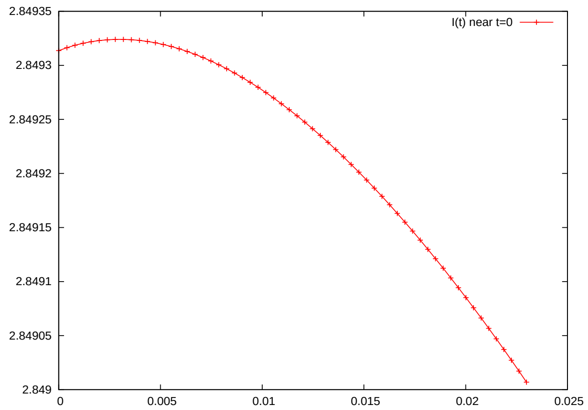


Figure 2: An example of behavior of $I(t)$ near $t = 0$

Next, we consider the ratio of $E(t)$ to $\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2$ to discuss another profile for asymptotic behavior near the blow-up time, where $E(t)$ is defined in (2.3) as follows:

$$E(t) = \int_\Omega u^{2-\delta}\,dx \int_\Omega u^{-\delta}(u_t)^2\,dx - \left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2 > 0.$$

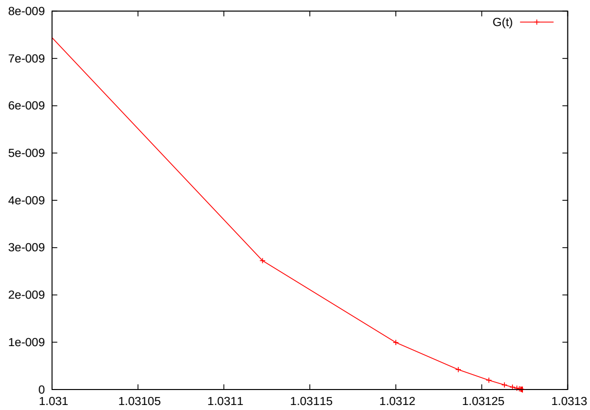
We set

$$F(t) := E(t)\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^{-2} = \frac{\int_\Omega u^{2-\delta}\,dx \int_\Omega u^{-\delta}(u_t)^2\,dx}{\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2} - 1$$

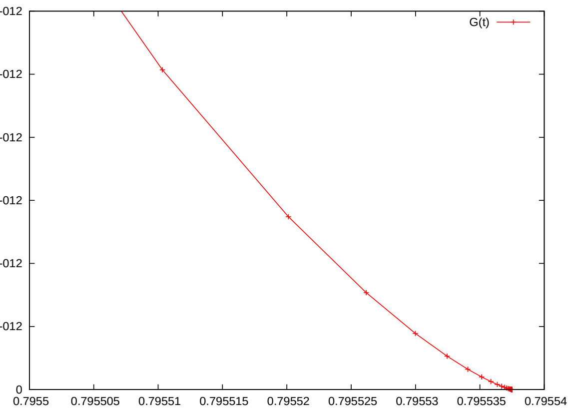
and

$$G(t) := F(t)^{-\frac{\delta}{\delta-2}}.$$

Then we numerically compute $G(t)$ with many kinds of initial data, too. Figure 3 (a) and (b) show the behavior of $G(t)$ near the blow-up time for the case of $\lambda = 1.0$, $\Omega = (0, 20)$ and the initial function (4.1).



(a) Behavior of $G(t)$ for $\delta = 2.5$



(b) Behavior of $G(t)$ for $\delta = 4.0$

Figure 3: Examples of behavior of $G(t) = F(t)^{-\frac{\delta}{\delta-2}}$

They indicate that $G(t)$ should decay as t closed to the blow-up time and the rate is faster than or equal to $O(T-t)$ as $t \nearrow T$. We have not found what does not satisfy this property, too.

From the above numerical observations, we would like to present a conjectures for behavior of $\int_\Omega u^{1-\delta}u_t\,dx$ and $F(t)$ as follows.

Conjectures. Let u be a solution of (1.2) approximated by u_ε solving (1.3) and T the blow-up time of u. When $\delta > 2$, there exists $t_0 \in (0, T)$, $C_1 > 0$ and $C_2 > 0$ such that

(C1) $\int_\Omega u^{1-\delta}u_t\,dx \ge C_1$ for $t \in [t_0, T)$.

(C2) $F(t) := E(t)\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^{-2}$ satisfies

$$F(t) \ge C_2(T-t)^{-\frac{\delta-2}{\delta}} \quad \text{for } t \in [t_0, T).$$

4.2. The Main Theorem for $\delta > 2$

If two conjectures, (C1) and (C2), are true then we can prove the following theorem which are similar to Theorem 1.

Theorem 2. *Let $\delta > 2$. u be a solution of (1.2) approximated by u_ε solving (1.3). uppose that $u_0 \in C^\infty(\Omega)\bigcap C(\overline{\Omega})$ satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$. If (C1) and (C2) hold then*

$$\delta(T-t)^{\frac{2}{\delta}} \int_\Omega u^{1-\delta} u_t\,dx \le \frac{1}{C_2}\int_\Omega u^{2-\delta}\,dx.$$

for $t \in [t_0, T)$, where $t_0 \in (0,T)$ and $C_2 > 0$ are given in Conjectures, (C1) and (C2).

Remark 6. If $u_0^{2-\delta} \in L^1(\Omega)$ then $u(\cdot,t)^{2-\delta} \in L^1(\Omega)$ for any $t \in (0,T)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be defined in Section 2. Then it is verified by Fatou's Lemma and (C2) that

$$\begin{aligned}
&\int_\Omega u^{2-\delta}\,dx \cdot \liminf_{n\to\infty}\int_\Omega u_n^{-\delta}\Big((u_n)_t\Big)^2\,dx \\
&= \int_\Omega u^{2-\delta}\,dx\cdot\liminf_{n\to\infty}\int_\Omega (\Delta u_n + \lambda u_n)(u_n)_t\,dx \\
&\quad\ge \int_\Omega u^{2-\delta}\,dx\int_\Omega (\Delta u + \lambda u)u_t\,dx \\
&\quad\ge \int_\Omega u^{2-\delta}\,dx\int_\Omega u^{-\delta}(u_t)^2\,dx \\
&\quad\ge \Big(C_2(T-t)^{-\frac{\delta-2}{\delta}}+1\Big)\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2
\end{aligned}$$

for $t \in [t_0,T)$. Besides,

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega u_n^{1-\delta}(u_n)_t\,dx &= \lim_{n\to\infty}\int_\Omega u_n(\Delta u_n+\lambda u_n)\,dx \\
&= \int_\Omega u(\Delta u + \lambda u)\,dx \\
&= \int_\Omega u^{1-\delta}u_t\,dx
\end{aligned}$$

because $u_n(\cdot,t) \to u(\cdot,t)$ uniformly in $\overline{\Omega}$ and $\Delta u_n \rightharpoonup \Delta u$ weakly in $C^*(\Omega)$ as $n\to\infty$ for any $t\in(0,T)$. Hence, for $\eta\in(0,C_1^2)$, there exists $N\in\mathbb{N}$ such that

$$\begin{aligned}
&\int_\Omega u^{2-\delta}\,dx\left(\int_\Omega u_n^{-\delta}\Big((u_n)_t\Big)^2\,dx + \eta\right) \\
&\ge \Big(C_2(T-t)^{-\frac{\delta-2}{\delta}}+1\Big)\left[\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)^2 - \eta\right] > 0
\end{aligned}$$

and $\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)^2 > \left(\int_\Omega u^{1-\delta}u_t\,dx\right)^2 - \eta > C_1^2 - \eta$ for $n \ge N$ and $t \in [t_0,T)$. Furthermore, by (2.4) and (C1), we have

$$\begin{aligned}
&\frac{2\eta}{C_1^2-\eta}\left(C_2(T-t)^{-\frac{\delta-2}{\delta}} + 1 + \int_\Omega u(x,t_0)^{2-\delta}\,dx\right) \\
&\qquad\ge 2\eta\left(C_2(T-t)^{-\frac{\delta-2}{\delta}} + 1 + \int_\Omega u(x,t)^{2-\delta}\,dx\right) \\
&\qquad\qquad\times\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)^{-2} \\
&\qquad\ge 2C_2(T-t)^{-\frac{\delta-2}{\delta}} \\
&\qquad\qquad+\int_\Omega u^{2-\delta}\,dx\left[\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)^{-1}\right]_t + 2
\end{aligned}$$

for $n \ge N$ and $t\in[t_0,T)$. It is verified by integral by parts that if $t_0 \le t < s < T$ and $\delta > 2$ then

$$\begin{aligned}
&\frac{\eta}{C_1^2-\eta}\Bigg[-\delta C_2(T-s)^{\frac{2}{\delta}} + \delta C(T-t)^{\frac{2}{\delta}} \\
&\qquad\qquad + 2\left(1+\int_\Omega u(x,t_0)^{2-\delta}\right)(s-t)\Bigg] \\
&\ge \delta C_2\left[-(T-s)^{\frac{2}{\delta}} + (T-t)^{\frac{2}{\delta}}\right] \\
&\quad + \int_\Omega u(x,s)^{2-\delta}\,dx\left(\int_\Omega u_n(x,s)^{1-\delta}(u_n)_t(x,s)\,dx\right)^{-1} \\
&\quad - \int_\Omega u(x,t)^{2-\delta}\,dx\left(\int_\Omega u_n(x,t)^{1-\delta}(u_n)_t(x,t)\,dx\right)^{-1} \\
&\quad + (\delta-2)(s-t) - \frac{\eta(\delta-2)}{\sqrt{C_1^2-\eta}}(s-t) + 2(s-t) \\
&\ge \delta C_2\left[-(T-s)^{\frac{2}{\delta}} + (T-t)^{\frac{2}{\delta}}\right] \\
&\quad - \int_\Omega u(x,t)^{2-\delta}\,dx\left(\int_\Omega u_n(x,t)^{1-\delta}(u_n)_t(x,t)\,dx\right)^{-1} \\
&\quad + \delta(s-t) - \frac{\eta(\delta-2)}{\sqrt{C_1^2-\eta}}(s-t).
\end{aligned}$$

Letting $s \nearrow T$, $n\to\infty$ and $\eta \searrow 0$, we have

$$\begin{aligned}
0 &\ge \delta C_2(T-t)^{\frac{2}{\delta}} - \int_\Omega u^{2-\delta}\,dx\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^{-1} \\
&\qquad + \delta(T-t) \\
&\ge \delta C_2(T-t)^{\frac{2}{\delta}} - \int_\Omega u^{2-\delta}\,dx\left(\int_\Omega u^{1-\delta}u_t\,dx\right)^{-1}
\end{aligned}$$

for $t\in[t_0,T)$. Therefore, it holds that

$$\delta(T-t)^{\frac{2}{\delta}}\int_\Omega u^{1-\delta}u_t\,dx \le \frac{1}{C_2}\int_\Omega u^{2-\delta}\,dx$$

which completes this proof. □

Moreover, we can also get the following corollary in the same manner of corollary 2.

Corollary 4. *Let $\delta > 2$ and u be a "Type 2" solution of (1.2) approximated by u_ε solving (1.3). Suppose that*

$u_0 \in C^\infty(\Omega)\bigcap C(\overline{\Omega})$ satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$. If (C1) and (C2) holds then there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and $U(x,t) := \dfrac{u(x,t)}{\|u(\cdot,t)\|_\infty}$ satisfies

$$\frac{\int_\Omega |\nabla U(x,t_k)|^2\,dx}{\int_\Omega U(x,t_k)^2\,dx} \to \lambda \quad \text{as } k \to \infty,$$

Proof. Since

$$\int_\Omega u^{1-\delta} u_t\,dx = \int_\Omega u(\Delta u + \lambda u)\,dx = \int_\Omega \Big(-|\nabla u|^2 + \lambda u^2 \Big)\,dx,$$

Theorem 2 implies

$$\int_\Omega u(\Delta u + \lambda u)\,dx \le \frac{1}{\delta C_2 (T-t)^{\frac{2}{\delta}}} \int_\Omega u^{2-\delta}\,dx$$

and then

$$\int_\Omega \Big(-|\nabla u(x,t)|^2 + \lambda u(x,t)^2 \Big)\,dx \le \frac{1}{\delta C_2 (T-t)^{\frac{2}{\delta}}} \int_\Omega u^{2-\delta}\,dx$$

for $t \in [t_0, T)$. Hence, $U(x,t) = \dfrac{u(x,t)}{\|u(\cdot,t)\|_\infty}$ satisfies

$$\lambda \int_\Omega U(x,t)^2\,dx - \frac{1}{\delta C_2 (T-t)^{\frac{2}{\delta}} \|u(\cdot,t)\|_\infty^2} \int_\Omega u^{2-\delta}\,dx \le \int_\Omega |\nabla U(x,t)|^2\,dx$$

for $t \in [t_0, T)$. Besides, (C1) implies $\int_\Omega u^{2-\delta}\,dx$ is decreasing in $[t_0, T)$ and

$$\int_\Omega |\nabla U(x,t)|^2\,dx < \lambda \int_\Omega U(x,t)^2\,dx \quad \text{for } t \in [t_0, T).$$

Since u is Type 2, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that

$$t_k \nearrow T \text{ and } (T-t_k)^{\frac{2}{\delta}} \|u(\cdot,t_k)\|_\infty^2 \to \infty \quad \text{as } k \to \infty.$$

Therefore, we have

$$\frac{\int_\Omega |\nabla U(x,t_k)|^2\,dx}{\int_\Omega U(x,t_k)^2\,dx} \to \lambda \quad \text{as } k \to \infty. \qquad \square$$

Remark 7. If u_0 satisfies $\Delta u_0 + \lambda u_0 \ge 0$ in Ω, then it also holds for $\delta > 2$ that there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and U satisfies

$$\Delta U(x,t_k) + \lambda U(x,t_k) \to 0 \text{ or } U(x,t_k) \to 0$$

as $k \to \infty$ almost all of $x \in \Omega$. We can regard Corollary 4 as a generalization of this profile.

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