Asymptotic behavior of blow-up solutions to a degenerate parabolic equation

Koichi Anada and Tetsuya Ishiwata

Revised on October 25, 2011

Abstract. We consider the Dirichlet problems for a degenerate parabolic equation, \( u_t = u^\delta(\Delta u + \lambda u) \) in a bounded domain in \( \mathbb{R}^n \) with a smooth boundary. It has been known that if \( \delta \geq 2 \) then there exists \( u \) which blows up faster than the rate of \( (T - t)^{-1/\delta} \), where \( T \) is the blow-up time of \( u \). The solutions are called “Type 2”. In this paper we investigate features for asymptotic behavior of “Type 2” solutions for the case of \( \delta = 2 \) and \( \delta > 2 \).

Keywords. degenerate parabolic equations, blow-up, asymptotic behavior, type 2, eventual monotonicity

1. INTRODUCTION

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). Suppose that a function \( u_0 \in C^\infty(\Omega) \cap C(\bar{\Omega}) \) satisfies

\[
    u_0 > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0(x) = 0 \quad \text{for} \quad x \in \partial \Omega. \tag{1.1}
\]

Then, we consider the following initial-boundary value problems,

\[
    \begin{cases}
        \frac{\partial u}{\partial t} = u^\delta(\Delta u + \lambda u) & x \in \Omega, \ t > 0, \\
        u(x, 0) = u_0(x) & x \in \Omega, \\
        u(x, t) = 0 & x \in \partial \Omega, \ t \geq 0,
    \end{cases} \tag{1.2}
\]

where \( \delta > 0 \) and \( \lambda \) is greater than the first eigenvalue, \( \lambda_1(\Omega) \) of \( -\Delta \) in \( \Omega \), that is,

\[
    \lambda > \lambda_1(\Omega).
\]

In this paper we discuss classical solutions which are approximated by functions \( u_\varepsilon \) solving the following problems:

\[
    \begin{cases}
        \frac{\partial u_\varepsilon}{\partial t} = u_\varepsilon^\delta(\Delta u_\varepsilon + \lambda u_\varepsilon) & x \in \Omega, \ t > 0, \\
        u_\varepsilon(x, 0) = u_0(x) + \varepsilon & x \in \Omega, \\
        u_\varepsilon(x, t) = \varepsilon & x \in \partial \Omega, \ t \geq 0.
    \end{cases} \tag{1.3}
\]

It has been proved that each of them blows up at a finite time, that is, for any solutions approximated by \( u_\varepsilon \), there exists \( T > 0 \) such that

\[
    \limsup_{t \nearrow T} \|u(\cdot, t)\|_\infty = \infty,
\]

where \( \|u(\cdot, t)\|_\infty = \sup_{x \in \Omega} u(x, t) \). (For instance, see [2].)

Here the constant \( T \) is called “the blow-up time” of \( u \). Moreover it has been known that they are classified into two types by their blow-up rates as follows.

Definition 1. Let \( u \) and \( T \) be a solution of (1.2) and the blow-up time of \( u \), respectively. Then \( u \) is called “Type 1” if it satisfies

\[
    \limsup_{t \nearrow T} (T - t)^{\frac{1}{\delta}} \|u(\cdot, t)\|_\infty < +\infty \tag{1.4}
\]

and \( u \) is called “Type 2” if it satisfies

\[
    \limsup_{t \nearrow T} (T - t)^{\frac{2}{\delta}} \|u(\cdot, t)\|_\infty = +\infty. \tag{1.5}
\]

Precisely, it has been proved that if \( \delta \geq 2 \) then there exists solutions of (1.2) which are “Type 2” and if \( 0 < \delta < 2 \) then \( u \) is “Type 1”. (See [1], [3], [4], [6], [9], [10] and so force.) Our purpose of this paper is to investigate asymptotic behavior of “Type 2” solutions for \( \delta \geq 2 \).

First, we consider the case of \( \delta = 2 \). S. Angenent gave important asymptotic profiles in [3] for curve shortening problems as follows.

\[
    v_t = v^2(v_{xx} + v) \quad \text{with periodic boundary conditions.}
\]

The first profile is that there exists \( t_0 > 0 \) such that \( v_t(x, t) = v(x, t)^2 \left( v_{xx}(x, t) + v(x, t) \right) > 0 \) for any \( x \) and \( t > t_0 \). This property is called “eventual monotonicity”.

Secondly, it was proved that \( V(x, t) := \frac{v(x, t)}{\|v(\cdot, t)\|_\infty} \) satisfies

\[
    V(x, t) \to \cos(x - x_0) \quad \text{if} \quad |x - x_0| \leq \frac{\pi}{2}, \tag{1.6}
\]

as \( t \nearrow T \) for some \( x_0 \). Besides, an upper bound for blow-up rates of solutions was given. In order to prove its existence, Angenent investigated relations between \( \int_\Omega \log v(x, t) \, dx \) and \( \log(T - t) \).
Their results hold under a special situation in curve shortening problems but we have not known whether they can be proved without any special assumptions.

In Section 3 we will discuss asymptotic behavior of solution to (1.2) without any special assumptions in the case of $\delta = 2$. Our method is based on behavior of $\int_\Omega u^{1-\delta} u_t \, dx$ as $t$ tends to the blow-up time of $u$. Precisely, we will prove in Section 3 that if $\delta = 2$ then $2(T - t) \int_\Omega u^{1-\delta} u_t \, dx$ becomes positive and has an upper bound after a finite time. In particular, the positivity implies that there exists $t_0 \in (0,T)$ such that $\int_\Omega u^{-1} u_t \, dx = \int_\Omega u(\Delta u + \lambda u) \, dx > 0$ for $t \in [t_0,T)$ and we call this property “weak eventual monotonicity”.

The positivity and the upper boundedness of $2(T - t) \int_\Omega u^{-1} u_t \, dx$ on $\delta = 2$ lead to two profiles for asymptotic behavior of solutions to (1.2). One is that there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $t_k \not\to T$ and
\[
\frac{\int_\Omega |\nabla U(x,t_k)|^2 \, dx}{\int_\Omega U(x,t_k) \, dx} \to \lambda \quad \text{as} \quad k \to \infty, \tag{1.7}
\]
where $U(x,t) := \frac{u(x,t)}{\|u(.t)||_\infty}$. We can regard (1.7) as a generalization of (1.6). Another is that if $\delta = 2$ and $\log u_0 \in L^1(\Omega)$ then $\int_\Omega \log [(T - t)^{\frac{\delta}{2}} u(x,t)] \, dx$ has an upper bound, that is, there exists a positive constant $C > 0$ such that
\[
\int_\Omega \log [(T - t)^{\frac{\delta}{2}} u(x,t)] \, dx \leq C \tag{1.8}
\]
for $t \in [t_0,T)$, where $C$ depends on $T$ and $u_0$ only. It may give a important profile for blow-up rates of “Type 2” that (1.5) holds for $\delta = 2$ but satisfies (1.8).

In Section 4 the case of $\delta > 2$ will be discussed. We expect that behavior of solutions for $\delta > 2$ may be a little different from $\delta = 2$. In the situation, we investigate features of $\int_\Omega u^{1-\delta} u_t \, dx$, $\int_\Omega u^{2-\delta} \, dx$ and others by numerical computing. And then we will present two conjectures for asymptotic behavior of solutions of (1.2) by numerical observations. Moreover, we will prove under their conjectures that $\delta(T - t)^{\frac{\delta}{2}} \int_\Omega u^{1-\delta} u_t \, dx$ has an upper bound after a finite time and (1.7) holds for some sequences.

## 2. Some Lemmas

Let $u$ be a solution of (1.2) approximated by $u_\varepsilon$ solving (1.3). The following two lemmas have been well-known in previous studies for asymptotic behavior of $u$.

**Lemma 1.** It holds that
\[ u_t(x,t) \geq -\frac{1}{\delta t} u(x,t) \quad \text{for} \quad x \in \Omega \quad \text{and} \quad t \in (0,T). \]

**Lemma 2.** If $\{t_k\}_{k \in \mathbb{N}}$ is a sequence such that $t_k \not\to T$ as $k \to \infty$, then there is $\mu \in (0,1)$ and a set $S_\mu \subset \Omega$ with positive Lebesgue measure such that
\[ u(x,t) \geq \mu \|u(\cdot,t)\|_\infty \quad \text{for} \quad x \in S_\mu \quad \text{and} \quad t \in (0,T) \]
for all $k$ from an appropriate subsequence. In particular, the Lebesgue measure of $\{x \in \Omega \mid \limsup_{t \to T} u(x,t) = \infty\}$ is positive.

They are very important properties for our purpose and can be proved by the maximum principle for parabolic equations. (For instance, see [9] and [10] for their proofs.) Moreover, we give another important property for $u$.

**Lemma 3.** If $u_0 \in C^\infty(\Omega) \cap C(\overline{\Omega})$ satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$ then
\[
\left( \int_\Omega u^{1-\delta} u_t \, dx \right)^2 < \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta}(u_t)^2 \, dx \tag{2.1}
\]
for any $t \in (0,T)$.

**Remark 1.** In Lemma 3, $u^{1-\delta} u_t$, $u^{-\delta}(u_t)^2$ and $u^{2-\delta}$ is integrable in $\Omega$ for any $t \in (0,T)$ because $u$ is a classical solution of $u_t = u^{\delta}(\Delta u + \lambda u)$.

**Remark 2.** When $\delta = 2$, the assumption $u_0^{2-\delta} \in L^1(\Omega)$ is not required and then we can prove
\[
\left( \int_\Omega u^{1-\delta} u_t \, dx \right)^2 < |\Omega| \int_\Omega u^{2-\delta}(u_t)^2 \, dx,
\]
where $|\Omega|$ is the Lebesgue measure of $\Omega$.

**Proof.** By Schwartz’s inequality,
\[
\left( \int_\Omega u^{1-\delta} u_t \, dx \right)^2 \leq \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta}(u_t)^2 \, dx. \tag{2.2}
\]
Note that $\int_\Omega u^{2-\delta} \, dx = |\Omega|$ if $\delta = 2$. Now, $u^{2-\delta} \not\equiv u^{-\delta}(u_t)^2$ because $u^{-1} u_t$ is not constant in $\Omega$. Hence, the equality in (2.2) does not hold and we get this lemma. \qed

**Remark 3.** (2.1) implies that When $\delta > 2$, if $u_0$ satisfies (1.1) and $u_0^{2-\delta} \in L^1(\Omega)$ then
\[
E(t) := \int_\Omega u^{2-\delta} \, dx \int_\Omega u^{-\delta}(u_t)^2 \, dx
- \left( \int_\Omega u^{1-\delta} u_t \, dx \right)^2 > 0. \tag{2.3}
\]
We will discuss the ratio of $E(t)$ to $(\int_{\Omega} u^{1-\delta} u_t dx)^2$ near the blow-up time in Section 4.

Let \( \{u_{0,n}\}_{n \in \mathbb{N}} \subset C^\infty(\Omega) \) be such that \( \|u_{0,n} - u_0\|_{L^\infty(\Omega)} < \frac{1}{k} \) and \( \alpha_n \Theta \leq u_{0,n} \leq \delta \Theta \) holds with constants \( 0 < \alpha_n < \delta \), where \( \Theta \) denotes the principal eigenfunction of \( -\Delta \) in \( \Omega \) with \( \max \Theta = 1 \). Then it has been shown in [9] that if \( \delta > 1 \) then the problems

\[
\begin{align*}
\frac{\partial u_n}{\partial t} &= u_n^\delta (\Delta u_n + \lambda u_n) \quad x \in \Omega, t \in (0, T(u_n)), \\
\varphi_n(x, 0) &= u_0(x) \quad x \in \overline{\Omega}, \\
\varphi_n(x, t) &= 0 \quad x \in \partial \Omega, t \geq 0,
\end{align*}
\]

have unique solutions \( u_n \in C(\overline{\Omega} \times [0, T(u_n))] \cap C^\infty(\Omega \times (0, T(u_n)), T(u_n) \to T, \) which may be approximated by function \( u_{n,\varepsilon} \) solving (1.3) with \( u_{n,\varepsilon} = u_n + \varepsilon \) on the parabolic boundary. As \( n \to \infty \), \( u_{n,\varepsilon} \to u \) in the topology of \( C_{loc}(\overline{\Omega} \times [0, T)) \cap C^\infty(\Omega \times (0, T)) \).

Furthermore, \( u_n \) have the property that for each \( n \in \mathbb{N} \) and each \( \tau > 0 \) there is a constant \( c(n, \tau) > 0 \) such that

\[ \text{dist} (x, \Omega)^{\alpha + 1} \frac{1}{2(\delta - 2)} \frac{\partial}{\partial t} u_n(x, t) \leq c(n, \tau) \]

for \( j = 0, 1, \alpha = 0, 1, 2 \) and \((x, t) \in \Omega \times (0, T - \tau) \). In particular, \( |\nabla(u_n)| \leq \text{dist} (x, \partial \Omega)^{\delta - 2} c(n, \tau) \) for \((x, t) \in \Omega \times (0, T - \tau) \) and \( u_n \) satisfies

\[
\int_{\Omega} u_n^{-\delta} (u_n)_t \ dx < \int_{\Omega} (\Delta u_n + \lambda u_n)(u_n)_t \ dx
\]

for \( t \in (0, T(u_n)) \).

Therefore, by Remark 2, we obtain the following lemma.

**Lemma 4.** If \( \delta = 2 \) then

\[
2 \left( \int_{\Omega} u_n^{-1}(u_n)_t \ dx \right)^2 < |\Omega| \left( \int_{\Omega} u_n^{-1}(u_n)_t \ dx \right)_t
\]

for any \( t \in (0, T(u_n)) \).

And Lemma 4 leads to the following corollary.

**Corollary 1.** If \( \delta = 2 \) then \( \int_{\Omega} u_n^{-1}(u_n)_t \ dx \) is strictly increasing with respect to \( t \).

**Proof.** By Lemma 4, if \( \delta = 2 \) then

\[
\left( \int_{\Omega} u_n^{-1}(u_n)_t \ dx \right)_t \geq \frac{2}{|\Omega|} \left( \int_{\Omega} u_n^{-1}(u_n)_t \ dx \right)^2 > 0
\]

for any \( t \in (0, T(u_n)) \).

3. **The Case of \( \delta = 2 \)**

3.1. **The Main Theorem for \( \delta = 2 \)**

We consider the case of \( \delta = 2 \), that is, the following problem.

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = u^2(\Delta u + \lambda u) & & (x, t) \in \Omega \times (0, T), \\
&u(x, 0) = u_0(x) & & x \in \overline{\Omega}, \\
&u(x, t) = 0 & & (x, t) \in \partial \Omega \times (0, T),
\end{aligned}
\]

where \( T \) is the blow-up time of \( u \). Then we can prove the following theorem.

**Theorem 1.** Let \( \delta = 2 \) and \( u \) be a solution approximated by \( u_{\varepsilon} \) solving (1.3). Suppose that \( u_0 \in C^\infty(\Omega) \cap C(\overline{\Omega}) \) satisfies (1.1) and \( \log u_0 \in L^1(\Omega) \). Then there exists \( t_0 > 0 \) such that

\[
0 < 2(T - t) \int_{\Omega} u^{1-\delta} u_t dx \leq |\Omega|
\]

for any \( t \in [t_0, T) \), where \( T > 0 \) is the blow-up time of \( u \).

**Remark 4.** The left side inequality in Theorem 1 implies that \( \int_{\Omega} u^{1-\delta} u_t dx = \int_{\Omega} (\Delta u + \lambda u) dx > 0 \) for \( t \in [t_0, T) \).

In this paper we call this property “weak eventual monotonicity.”

**Proof of Theorem 1.** Let \( \tau > 0 \) be small enough and \( \{t_k\}_{k=1,2, \ldots} \) be a sequence with \( t_k > \tau, t_k \not\to T \) and \( \|u(\cdot, t_k)\|_{L^\infty} \not\to \infty \) as \( k \to \infty \). Since Lemma 1 implies \( (\log u)_t \geq -\frac{1}{2t} \), it holds that

\[
\log u(x, t_k) - \log u(x, \tau) \geq -\frac{1}{2} \log t_k - \log \tau > -\frac{1}{2} \log \frac{T}{\tau}.
\]

By Lemma 2, for an appropriate subsequence, there exists \( \mu \in (0, 1) \) and a set \( S_\mu \) such that \( |S_\mu| > 0 \) and

\[
\int_{\Omega} \log u(x, t_k) dx \geq \int_{\Omega \setminus S_\mu} \log u(x, t_k) dx + \int_{S_\mu} \log u(x, t_k) dx
\]

\[
\geq |S_\mu| \log \|u(\cdot, t_k)\|_{L^\infty} - \frac{1}{2} |\Omega \setminus S_\mu| \log \frac{T}{\tau} + \int_{S_\mu} \log u(x, \tau) dx.
\]

Since the last term is bounded for a fixed positive number \( \tau \in (0, T) \) because of \( \log u_0 \in L^1(\Omega) \), we can get

\[
\int_{\Omega} \log u(x, t_k) dx \to \infty \quad \text{as} \quad k \to \infty \quad \text{for any} \quad n \in \mathbb{N}
\]

which implies that there exists \( t_0 \) such that

\[
\int_{\Omega} u(x, t_0)^{-1} u_t(x, t_0) dx = \frac{d}{dt} \int_{\Omega} \log u(x, t_0) dx > 0.
\]
Hence, by Corollary 1,
\[ \int_{\Omega} u^{-1} u_t \, dx > 0 \quad \text{for any } t \in [t_0, T). \]

On the other hands, when \( \delta = 2 \), Lemma 4 leads to
\[ \frac{2(s - t)}{|\Omega|} + \left( \int_{\Omega} u_n(x, t)^{-1}(u_n)_t(x, s) \, dx \right)^{-1} \leq 0 \quad \text{for } t \in (0, T(u_n)) \]
for \( u_n \) defined in Section 2. Integrating from \( t \) to \( s \), we have
\[ \frac{2(s - t)}{|\Omega|} + \left( \int_{\Omega} u_n(x, s)^{-1}(u_n)_t(x, s) \, dx \right)^{-1} \leq 0. \]
Since
\[ \int_{\Omega} u_n(x, s)^{-1}(u_n)_t(x, s) \, dx \]
\[ > \int_{\Omega} u_n(x, t)^{-1}(u_n)_t(x, t) \, dx > 0 \]
if \( t_0 \leq t < s < T(u_n) \), it holds that
\[ \frac{2(s - t)}{|\Omega|} + \left( \int_{\Omega} u_n(x, t)^{-1}(u_n)_t(x, t) \, dx \right)^{-1} < 0 \]
where \( t_0 \leq t < s < T(u_n) \). Then it is verified by letting \( s / T(u_n) \) that
\[ 2(T(u_n) - t) \int_{\Omega} u_n(\Delta u_n + \lambda u_n) \, dx \]
\[ = 2(T(u_n) - t) \int_{\Omega} u_n^{-1}(u_n)_t \, dx \leq |\Omega| \]
for any \( t \in [t_0, T(u_n)) \).

Now, we mentioned in Section 2 that \( T(u_n) \to T \) and \( u_n \to u \) in the topology of \( C^\infty_0(\Omega \times [0, T]) \cap C^\infty_0(\Omega \times (0, T)) \) as \( n \to \infty \). This implies that \( u_n(s, t) \to u(s, t) \) uniformly in \( \Omega \) and \( \Delta u_n(s, t) \to \Delta u(s, t) \) weakly in \( C^* \) as \( n \to \infty \) for any \( t \in [t_0, T) \). Then we have
\[ 2(T - t) \int_{\Omega} u^{-1} u_t \, dx \]
\[ = 2(T - t) \int_{\Omega} u(\Delta u + \lambda u) \, dx \leq |\Omega| \]
which completes this proof.

3.2. Behavior near the Blow-up Time

We discuss some important profiles for asymptotic behavior of \( U(x, t) := \frac{u(x, t)}{\|u(\cdot, t)\|_\infty} \), where \( u \) is a “Type 2” blow-up solution to (3.1).

First, suppose that \( u_0 \) satisfies (1.1) and \( \Delta u_0 + \lambda u_0 \geq 0 \) in \( \Omega \). Then it is easily verified by the maximum principle that \( u > 0 \), \( \Delta u + \lambda u \geq 0 \) and \( U(\Delta U + \lambda U) \geq 0 \) in \( \Omega \times (0, T) \). Besides, Theorem 1 leads to
\[ 0 < \int_{\Omega} U(\Delta U + \lambda U) \, dx = \frac{1}{\|u(\cdot, t)\|_\infty^2} \int_{\Omega} u(\Delta u + \lambda u) \, dx \]
\[ = \frac{1}{\|u(\cdot, t)\|_\infty^2} \int_{\Omega} u^{-1} u_t \, dx \]
\[ \leq \frac{2(T - t)}{\|u(\cdot, t)\|_\infty^2} \]
for \( t \in [t_0, T) \). In addition, “Type 2” means
\[ \limsup_{t \to T} u(x, t) = +\infty. \]
Therefore, if \( u \) is “Type 2” and \( u_0 \) satisfies \( \Delta u_0 + \lambda u_0 \geq 0 \)

Corollary 2. Let \( \delta = 2 \) and \( u \) be a “Type 2” solution of (3.1) approximated by \( u_n \) solving (1.3). Suppose that \( u_0 \in C^\infty(\Omega) \cap C(\Omega) \) satisfies (1.1) and log \( u_0 \in L^1(\Omega) \). Then there exists a sequence \( \{t_k\}_{k \in \mathbb{N}} \) such that \( t_k \to T \) as \( k \to \infty \) and \( U(x, t) := \frac{u(x, t)}{\|u(\cdot, t)\|_\infty} \) satisfies
\[ \frac{\int_{\Omega} |
abla U(x, t_k)|^2 \, dx}{\int_{\Omega} U(x, t_k)^2 \, dx} \to \lambda \quad \text{as } k \to \infty. \]

Proof. If \( \delta = 2 \), then
\[ \int_{\Omega} u^{-1} u_t \, dx = \int_{\Omega} u(\Delta u + \lambda u) \, dx \]
\[ = \int_{\Omega} \left( -|\nabla u|^2 + \lambda u^2 \right) \, dx. \]
By Theorem 1, we have
\[ 0 < \int_{\Omega} \left( -|\nabla u(x, t)|^2 + \lambda u(x, t)^2 \right) \, dx \leq \frac{|\Omega|}{2(T - t)} \]
for \( t \in [t_0, T) \). Hence, \( U(x, t) = \frac{u(x, t)}{\|u(\cdot, t)\|_\infty} \) satisfies
\[ \frac{\int_{\Omega} |
abla U(x, t)|^2 \, dx}{\int_{\Omega} U(x, t)^2 \, dx} < \lambda \int_{\Omega} U(x, t)^2 \, dx \quad \text{(3.2)} \]
and
\[ \lambda \int_{\Omega} U(x, t)^2 \, dx = \frac{|\Omega|}{2(T - t)} \|u(\cdot, t)\|_\infty^2 \leq \int_{\Omega} |
abla U(x, t)|^2 \, dx. \quad \text{(3.3)} \]
Since if \( \delta = 2 \) and \( u \) is Type 2, then
\[
\limsup_{t \to T} (T - t)^{\frac{\delta}{2}} \|u(\cdot, t)\|_{\infty} = \infty,
\]
there exists a sequence \( \{t_k\}_{k \in \mathbb{N}} \) such that
\[
t_k \not\to T \quad \text{and} \quad (T - t_k)^{\frac{\delta}{2}} \|u(\cdot, t_k)\|_{\infty} \to \infty \quad \text{as} \quad k \to \infty.
\]
Furthermore, Lemma 2 implies that
\[
0 < \mu^2|S_\mu| \leq \int_{\Omega} U(x, t_k)^2 \, dx \leq |\Omega| \quad \text{for any} \quad k.
\tag{3.4}
\]
Therefore, we have
\[
\int_{\Omega} |\nabla U(x, t_k)|^2 \, dx \to \lambda \quad \text{as} \quad k \to \infty.
\]

3.3. A Profile for blow-up rates

We mentioned in Section 1 that there exists “Type 2” solutions of (3.1) for \( \delta = 2 \) which satisfies
\[
\limsup_{t \to T} (T - t)^{\frac{\delta}{2}} \|u(\cdot, t)\|_{\infty} = \infty \tag{3.5}
\]
The following corollary gives another feature for asymptotic behavior of \((T - t)^{\frac{\delta}{2}} u(x, t)\) as \( t \not\to T \).

**Corollary 3.** Let \( \delta = 2 \) and \( u \) be a solution of (3.1) approximated by \( u_\varepsilon \) solving (1.3). Suppose that \( u_0 \in C^\infty(\Omega) \cap C(\overline{\Omega}) \) satisfies (1.1) and \( \log u_0 \in L^1(\Omega) \). Then there exists a constant \( C > 0 \) such that
\[
\int_{\Omega} \log \left[(T - t)^{\frac{\delta}{2}} u(x, t)\right] \, dx \leq C
\]
for \( t \in [t_0, T) \), where \( C \) depends on \( T \) and \( u_0 \) only.

**Proof.** Since Theorem 1 implies
\[
0 < \frac{d}{dt} \int_{\Omega} \log u \, dx \leq \frac{|\Omega|}{2(T - t)} \quad \text{for any} \quad t \in [t_0, T),
\]
we have
\[
\frac{d}{dt} \int_{\Omega} \log \left[(T - t)^{\frac{\delta}{2}} u(x, t)\right] \, dx \leq 0 \quad \text{for any} \quad t \in [t_0, T).
\]
Therefore,
\[
\int_{\Omega} \log \left[(T - t)^{\frac{\delta}{2}} u(x, t)\right] \, dx \leq \int_{\Omega} \log \left[T^{\frac{\delta}{2}} u_0(x)\right] \, dx
\]
which completes this proof.

4. The Case of \( \delta > 2 \)

4.1. Conjectures for \( \delta > 2 \)

For the case of \( \delta > 2 \), we expect that behavior of blow-up solutions may be a little different from \( \delta = 2 \). In this section we try to establish some profiles for \( \delta > 2 \) by numerical computing.

Precisely, we investigated features for asymptotic behavior of blow-up solutions to (1.2) by numerical computing with many kinds of initial data. Note that all of our computations are based on the scheme provided in [5].

For example, we consider the case of \( \lambda = 1.0, \quad \Omega = (0, 20) \in \mathbb{R}^1 \) and
\[
u_0(x) = \frac{1}{100} \left( \sin \frac{\pi}{20} x \right) \left( \sin^2 \frac{3\pi}{20} x \right).
\tag{4.1}
\]

Figure 1 (a) and (b) are behavior of
\[
I(t) := \int_{\Omega} u^{2-\delta} \, dx
\]
for \( \delta = 2.5 \) and \( \delta = 4.0 \) near the blow-up time, respectively.

![Figure 1: Examples of behavior of \( I(t) = \int_{\Omega} u^{2-\delta} \, dx \)](image-url)
They show that $I(t)$ is decreasing near the blow-up time. Since \( \frac{d}{dt} I(t) = (2 - \delta) \int_{\Omega} u^{1-\delta} u_t \, dx \), this observation implies that if $\delta > 2$ then $\int_{\Omega} u^{1-\delta} u_t \, dx$ should be positive near the blow-up time, that is, $u$ should satisfy the same property as “weak eventual monotonicity” in Remark 4 for the case of $\delta = 2$. We are computing numerically with so many initial data but we have not gotten what does not satisfy “weak eventual monotonicity”.

**Remark 5.** The property of “weak eventual monotonicity” means that $\int_{\Omega} u^{1-\delta} u_t \, dx$ should change positive even if it is negative near the initial time $t = 0$, that is, $I(t) = \int_{\Omega} u^{2-\delta} \, dx$ should be strictly decreasing after a finite time. For instance, Figure 2 is an example of behavior near the initial time of $I(t) = \int_{\Omega} u^{2-\delta} \, dx$ with the initial data $u_0(x) = K^2 - x^2$, where $\Omega = (-K, K) \in \mathbb{R}$, $\delta = 2.5$ and $K = 1.575$.

![Figure 2: An example of behavior of $I(t)$ near $t = 0$](image)

Next, we consider the ratio of $E(t)$ to $\left( \int_{\Omega} u^{1-\delta} u_t \, dx \right)^2$ to discuss another profile for asymptotic behavior near the blow-up time, where $E(t)$ is defined in (2.3) as follows:

$$E(t) = \int_{\Omega} u^{2-\delta} \, dx \int_{\Omega} u^{-\delta} (u_t)^2 \, dx - \left( \int_{\Omega} u^{1-\delta} u_t \, dx \right)^2 > 0.$$  

We set

$$F(t) : = E(t) \left( \int_{\Omega} u^{1-\delta} u_t \, dx \right)^{-2}$$

$$= \int_{\Omega} u^{2-\delta} \, dx \int_{\Omega} u^{-\delta} (u_t)^2 \, dx \over \left( \int_{\Omega} u^{1-\delta} u_t \, dx \right)^2 - 1$$

and

$$G(t) : = F(t)^{-1 \over 2}.$$  

Then we numerically compute $G(t)$ with many kinds of initial data, too. Figure 3 (a) and (b) show the behavior of $G(t)$ near the blow-up time for the case of $\lambda = 1.0$, $\Omega = (0, 20)$ and the initial function (4.1).

![Figure 3: Examples of behavior of $G(t) = F(t)^{-1 \over 2}$](image)

They indicate that $G(t)$ should decay as $t$ closed to the blow-up time and the rate is faster than or equal to $O(T-t)$ as $t \nearrow T$. We have not found what does not satisfy this property, too.

From the above numerical observations, we would like to present a conjectures for behavior of $\int_{\Omega} u^{1-\delta} u_t \, dx$ and $F(t)$ as follows.

**Conjectures.** Let $u$ be a solution of (1.2) approximated by $u_\varepsilon$ solving (1.3) and $T$ the blow-up time of $u$. When $\delta > 2$, there exists $t_0 \in (0, T)$, $C_1 > 0$ and $C_2 > 0$ such that

(C1) $\int_{\Omega} u^{1-\delta} u_t \, dx \geq C_1$ for $t \in [t_0, T]$.

(C2) $F(t) := E(t) \left( \int_{\Omega} u^{1-\delta} u_t \, dx \right)^{-2}$ satisfies

$$F(t) \geq C_2 (T-t)^{-1 \over 2}$$

for $t \in [t_0, T]$. 
4.2. The Main Theorem for $\delta > 2$

If two conjectures, (C1) and (C2), are true then we can prove the following theorem which are similar to Theorem 1.

**Theorem 2.** Let $\delta > 2$, $u$ be a solution of (1.2) approximated by $u_\varepsilon$ solving (1.3). Suppose that $u_0 \in C^\infty(\Omega) \cap C(\overline{\Omega})$ satisfies (1.1) and $u_0^{\delta - 2} \in L^1(\Omega)$. If (C1) and (C2) hold then

$$
\delta(T - t)^{\frac{\delta}{\delta - 2}} \int_\Omega u^{\delta - 2} u_t dx \leq \frac{1}{C_2} \int_\Omega u^{\delta - 2} dx.
$$

for $t \in [t_0, T)$, where $t_0 \in (0, T)$ and $C_2 > 0$ are given in Conjectures, (C1) and (C2).

**Remark 6.** If $u_0^{\delta - 2} \in L^1(\Omega)$ then $u(\cdot, t)^{\delta - 2} \in L^1(\Omega)$ for any $t \in (0, T)$.

**Proof.** Let $\{u_n\}_{n \in \mathbb{N}}$ be defined in Section 2. Then it is verified by Fatou’s Lemma and (C2) that

$$
\int_\Omega u^{\delta - 2} dx \cdot \liminf_{n \to \infty} \int_\Omega u^{\delta - 2}(u_n)_t^2 dx
= \int_\Omega u^{\delta - 2} dx \cdot \liminf_{n \to \infty} \int_\Omega (\Delta u_n + \lambda u_n)(u_n)_t dx
\geq \int_\Omega u^{\delta - 2} dx \int_\Omega (\Delta u + \lambda u) u_t dx
\geq \int_\Omega u^{\delta - 2} dx \int_\Omega u_t dx
\geq \left(C_2(T - t)^{\frac{\delta}{\delta - 2}} + 1\right) \left(\int_\Omega u^{\delta - 2} u_t dx\right)^2
$$

for $t \in [t_0, T)$. Besides,

$$
\lim_{n \to \infty} \int_\Omega u^{\delta - 2}(u_n)_t dx = \lim_{n \to \infty} \int_\Omega u_n(\Delta u_n + \lambda u_n) dx
= \int_\Omega u(\Delta u + \lambda u) dx
= \int_\Omega u^{\delta - 2} u_t dx
$$

because $u_n(\cdot, t) \to u(\cdot, t)$ uniformly in $\overline{\Omega}$ and $\Delta u_n \to \Delta u$ weakly in $C^* \cap (\Omega)$ as $n \to \infty$ for any $t \in (0, T)$. Hence, for $\eta \in (0, C_2^{-1})$, there exists $N \in \mathbb{N}$ such that

$$
\int_\Omega u^{\delta - 2} dx \left(\int_\Omega u^{\delta - 2}(u_n)_t^2 dx + \eta\right)
\geq \left(C_2(T - t)^{\frac{\delta}{\delta - 2}} + 1\right) \left[\left(\int_\Omega u^{\delta - 2}(u_n)_t dx\right)^2 - \eta\right] > 0
$$

and

$$
\left(\int_\Omega u^{\delta - 2}(u_n)_t dx\right)^2 > \left(\int_\Omega u^{\delta - 2} u_t dx\right)^2 - \eta > C_2^2 - \eta
$$

for $n \geq N$ and $t \in [t_0, T)$. Furthermore, by (2.4) and (C1), we have

$$
\frac{2\eta}{C_2^2 - \eta} \left(C_2(T - t)^{\frac{\delta}{\delta - 2}} + 1 + \int_\Omega u(x, t_0)^{2 - \delta} dx\right)
\geq 2\eta \left(C_2(T - t)^{\frac{\delta}{\delta - 2}} + 1 + \int_\Omega u(x, t)^{2 - \delta} dx\right)
\times \left(\int_\Omega u^{\delta - 2}(u_n)_t dx\right)^{-2}
\geq 2C_2(T - t)^{\frac{\delta}{\delta - 2}} + \int_\Omega u^{\delta - 2} dx \left[\left(\int_\Omega u^{\delta - 2}(u_n)_t dx\right)^{-1}\right]_t^2
$$

for $n \geq N$ and $t \in [t_0, T)$. It is verified by integral by parts that if $t_0 \leq t < s < T$ and $\delta > 2$ then

$$
\frac{\eta}{C_2^2 - \eta} \left[-\delta C_2(T - s)^{\frac{\delta}{\delta - 2}} + \delta C(T - t)^{\frac{\delta}{\delta - 2}}\right]
+ 2 \left(1 + \int_\Omega u(x, t_0)^{2 - \delta}\right) (s - t)
\geq \delta C_2 \left[-(T - s)^{\frac{\delta}{\delta - 2}} + (T - t)^{\frac{\delta}{\delta - 2}}\right]
+ \int_\Omega u(x, s)^{2 - \delta} dx \left(\int_\Omega u_n(x, s)^{1 - \delta}(u_n)_t(x, s) dx\right)^{-1}
- \int_\Omega u(x, t)^{2 - \delta} dx \left(\int_\Omega u_n(x, t)^{1 - \delta}(u_n)_t(x, t) dx\right)^{-1}
+ (\delta - 2)(s - t) - \frac{\eta(\delta - 2)}{\sqrt{C_2^2 - \eta}}(s - t) + 2(s - t)
\geq \delta C_2 \left[-(T - s)^{\frac{\delta}{\delta - 2}} + (T - t)^{\frac{\delta}{\delta - 2}}\right]
- \int_\Omega u(x, s)^{2 - \delta} dx \left(\int_\Omega u_n(x, s)^{1 - \delta}(u_n)_t(x, s) dx\right)^{-1}
+ \delta(s - t) - \frac{\eta(\delta - 2)}{\sqrt{C_2^2 - \eta}}(s - t).
$$

Letting $s \not\to T$, $n \to \infty$ and $\eta \searrow 0$, we have

$$
0 \geq \delta C_2(T - t)^{\frac{\delta}{\delta - 2}} - \int_\Omega u^{2 - \delta} dx \left(\int_\Omega u^{\delta - 2} u_t dx\right)^{-1}
+ \delta(T - t)
\geq \delta C_2(T - t)^{\frac{\delta}{\delta - 2}} - \int_\Omega u^{2 - \delta} dx \left(\int_\Omega u^{\delta - 2} u_t dx\right)^{-1}
$$

for $t \in [t_0, T)$. Therefore, it holds that

$$
\delta(T - t)^{\frac{\delta}{\delta - 2}} \int_\Omega u^{\delta - 2} u_t dx \leq \frac{1}{C_2} \int_\Omega u^{2 - \delta} dx
$$

which completes this proof. \(\square\)

Moreover, we can also get the following corollary in the same manner of corollary 2.

**Corollary 4.** Let $\delta > 2$ and $u$ be a “Type 2” solution of (1.2) approximated by $u_\varepsilon$ solving (1.3). Suppose that
$u_0 \in C^\infty(\Omega) \cap C(\bar{\Omega})$ satisfies (1.1) and $u_0^\alpha - \delta \in L^1(\Omega)$. If (C1) and (C2) holds then there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and $U(x, t) := \frac{u(x, t)}{\|u(\cdot, t)\|_\infty}$ satisfies

$$\int_\Omega \frac{|\nabla U(x, t_k)|^2}{U(x, t_k)^2} \, dx \to \lambda \quad \text{as} \quad k \to \infty,$$

Proof. Since

$$\int_\Omega u^{1-\delta} \, dx = \int_\Omega u(\Delta u + \lambda u) \, dx = \int_\Omega (-|\nabla u|^2 + \lambda u^2) \, dx,$$

Theorem 2 implies

$$\int_\Omega u(\Delta u + \lambda u) \, dx \leq \frac{1}{\delta C_2(T-t)^\frac{\delta}{2}} \int_\Omega u^{2-\delta} \, dx$$

and then

$$\int_\Omega \left( - |\nabla u(x, t)|^2 + \lambda u(x, t)^2 \right) \, dx \leq \frac{1}{\delta C_2(T-t)^\frac{\delta}{2}} \int_\Omega u^{2-\delta} \, dx$$

for $t \in [t_0, T)$. Hence, $U(x, t) = \frac{u(x, t)}{\|u(\cdot, t)\|_\infty}$ satisfies

$$\lambda \int_\Omega U(x, t)^2 \, dx - \frac{1}{\delta C_2(T-t)^\frac{\delta}{2}} \int_\Omega u^{2-\delta} \, dx$$

$$\leq \int_\Omega |\nabla U(x, t)|^2 \, dx$$

for $t \in [t_0, T)$. Besides, (C1) implies $\int_\Omega u^{2-\delta} \, dx$ is decreasing in $[t_0, T)$ and

$$\int_\Omega |\nabla U(x, t)|^2 \, dx < \lambda \int_\Omega U(x, t)^2 \, dx \quad \text{for} \quad t \in [t_0, T).$$

Since $u$ is Type 2, there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that

$t_k \nearrow T$ and $(T-t_k)^\frac{\delta}{2} \|u(\cdot, t_k)\|_\infty \to \infty$ as $k \to \infty$.

Therefore, we have

$$\int_\Omega \frac{|\nabla U(x, t_k)|^2}{U(x, t_k)^2} \, dx \to \lambda \quad \text{as} \quad k \to \infty.$$

\textbf{Remark 7.} If $u_0$ satisfies $\Delta u_0 + \lambda u_0 \geq 0$ in $\Omega$, then it also holds for $\delta > 2$ that there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $t_k \nearrow T$ as $k \to \infty$ and $U$ satisfies

$$\Delta U(x, t_k) + \lambda U(x, t_k) \to 0 \quad \text{or} \quad U(x, t_k) \to 0$$

as $k \to \infty$ almost all of $x \in \Omega$. We can regard Corollary 4 as a generalization of this profile.

\textbf{Acknowledgements}

We would like to thank referees very much for their kind comments and valuable suggestions to improve our paper.

\textbf{References}


Koichi Anada  
Waseda University Senior High School, 3-31-1 Kamishakujii Nerima-ku, Tokyo 177-0044, JAPAN.  
E-mail: anada-koichi(at)waseda.jp

Tetsuya Ishiwata  
Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama 337-8570, JAPAN.  
E-mail: tisiwata(at)shibaura-it.ac.jp