Journal of Math-for-Industry, Vol. 3 (2011B-5), pp. 119–123

Differential-geometric structures of ideal magnetohydrodynamics and plasma instabilities

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Received on August 11, 2011 / Revised on September 27, 2011

Abstract. The differential-geometric structures of ideal magnetohydrodynamics are studied by calculating sectional curvatures of a semidirect product of the volume-preserving diffeomorphism group and the vector field on it. Some propositions on the negativeness and positiveness of the sectional curvatures are derived. In particular, the curvature of the section corresponding to the sausage instability of plasma is shown to be negative.

Keywords. ideal magnetohydrodynamics, differential geometry, sectional curvature, instability

1. Introduction

It is well known that the equations of motion of ideal (ideal implies inviscid and incompressible throughout this paper) hydrodynamics (HD) and ideal magnetohydrodynamics (MHD) are infinite-dimensional Hamiltonian systems, or more specifically, Lie-Poisson systems [2, 3, 4, 12, 13, 17]. Moreover, both ideal HD and ideal MHD are regarded as geodesic flows on infinite-dimensional Lie groups when we recall that a Lie-Poisson system whose Hamiltonian is quadratic is an equation of the geodesic flow on the corresponding Lie group with one-sided invariant metric as shown by Arnol'd [2, 3, 4]. The result by Arnol'd was extended to the case of Lie-Poisson systems with non-quadratic Hamiltonian by Ono [15].

The Hamiltonian formulations of ideal HD and ideal MHD are quite useful in deriving important results such as theorems on *nonlinear* stability, conservation laws, and so on [2, 3, 8, 9]. In contrast to the Hamiltonian formulation, the differential-geometric formulation is not often used in the analysis of fluid motion. Actually, the differential-geometric structures of ideal HD have been studied by calculating sectional curvatures of the group [2, 3, 10, 11, 14, 18]. We have tried to give a reasonable basis for exponential stretching of line elements in turbulence using the differential-geometric formulation [7]. Much attention has been also given to the differential-geometric formulation of ideal MHD [6, 16, 19]. However, the differential-geometric structures of ideal MHD are not fully understood; the curvature tensors are explicitly obtained in the limited case of periodic boundary conditions [16, 19]; we have little knowledge of the properties of sectional curvatures.

The sectional curvatures are of much interest, since they are closely related with the motion of geodesics [2, 3, 4]. Negativeness of sectional curvature usually implies exponential instability of geodesics and positiveness implies neutral stability. In particular, when the sectional curvature is always negative, the geodesic flow is ergodic [1].

In this paper, we study the differential-geometric structures of ideal MHD by showing some explicit forms of sectional curvatures for a general flow domain. Some propositions on the sign of sectional curvatures are derived. As a step to the practical applications of the differential-geometric formulation, the curvature for a practical situation corresponding to the sausage instability, which is one of the fundamental instabilities of plasma, is shown to be negative in accordance with the physical phenomena. This result strongly supports that the differential-geometric formulation of hydrodynamics and magnetohydrodynamics can be a tool for analyzing practical phenomena including those encountered in the industries.

2. Differential-geometric formulation of ideal MHD

In this section, we summarize the differential-geometric formulation of ideal MHD in general dimensions for later use.

Let $\mathcal{D}_v(M)$ be a group of volume-preserving diffeomorphisms on M and $\mathcal{X}_0(M)$ a space of divergenceless vector fields on M. For simplicity, the N-dimensional domain $M \in \mathbf{R}^N$ of MHD flow is assumed to be a simply connected finite region or a flat torus with periodic boundary. Let G be a semidirect product of $\mathcal{D}_v(M)$ and $\mathcal{X}_0(M)$ with multiplication

$$(g,\gamma)\circ(h,\eta) = (g\circ h, \mathrm{Ad}_{h^{-1}}\gamma + \eta), \tag{1}$$

for $g, h \in \mathcal{D}_v(M), \gamma, \eta \in \mathcal{X}_0(M)$, where $\mathrm{Ad}_{h^{-1}} = \tilde{L}_{h^{-1}}\tilde{R}_h$ is the adjoint action and $\tilde{L}_h$ and $\tilde{R}_h$ are the differentials of left and right translations, respectively.

The Lie algebra of G is defined as the linear space $\mathcal{X}^R(G)$ of all right-invariant vector fields on G, that is, $\mathcal{X}^R(G) =$

$\{(u,\alpha)^R; (u,\alpha)^R|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)}(u,\alpha),\ (u,\alpha) \in T_{(e,0)}G = \mathcal{X}_0(M) \times \mathcal{X}_0(M)\}$.

We construct a right-invariant metric on G by defining it at the identity of G and extending it to each point by right translation. That is,

$$\begin{aligned}
&\langle (u,\alpha),(v,\beta)\rangle|_{(e,0)} \\
&= \int_M u \cdot v \, \mathrm{d}^N x + \int_M \alpha \cdot (-\Delta^{-1})\beta \, \mathrm{d}^N x \\
&= \int_M \sum_{i=1}^{N} \left(u_i v_i + \alpha_i (-\Delta^{-1})\beta_i \right) \, \mathrm{d}^N x
\end{aligned} \tag{2}$$

for $(u,\alpha),(v,\beta) \in T_{(e,0)}G$ ($\Delta = \nabla \cdot \nabla$, $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_N)$) and

$$\begin{aligned}
&\langle (u',\alpha'),(v',\beta')\rangle|_{(h,\gamma)} \\
&= \langle \tilde{R}_{(h^{-1},-\mathrm{Ad}_h\gamma)}(u',\alpha'), \tilde{R}_{(h^{-1},-\mathrm{Ad}_h\gamma)}(v',\beta')\rangle|_{(e,0)}
\end{aligned}$$

for $(u',\alpha'),(v',\beta') \in T_{(h,\gamma)}G$. If we regard u as a velocity field and α as a current field, the metric defined above is related with the total energy of MHD flow by $2E_T = \langle (u,\alpha),(u,\alpha)\rangle|_{(e,0)}$. The Levi-Civita connection $\tilde{\nabla}$ for this metric is expressed for right-invariant vector fields as

$$\begin{aligned}
&\tilde{\nabla}_{(u,\alpha)^R}(v,\beta)^R|_{(h,\gamma)} \\
&= \tilde{R}_{(h,\gamma)} \Big(\ P[(u\cdot\nabla)v - \frac{1}{2}(\partial_i \alpha_j \Delta^{-1}\beta_j + \Delta^{-1}\alpha_j \partial_i \beta_j \\
&\qquad + (\alpha\cdot\nabla)\Delta^{-1}\beta + (\beta\cdot\nabla)\Delta^{-1}\alpha)], \\
&\quad \frac{1}{2} P\Big\{ \Delta \left[\partial_i u_j \Delta^{-1}\beta_j + \Delta^{-1}\alpha_j \partial_i v_j \right. \\
&\qquad \left. +(u\cdot\nabla)\Delta^{-1}\beta + (v\cdot\nabla)\Delta^{-1}\alpha \right] \\
&\qquad + (u\cdot\nabla)\beta - (\beta\cdot\nabla)u \\
&\qquad + (\alpha\cdot\nabla)v - (v\cdot\nabla)\alpha \Big\} \Big),
\end{aligned} \tag{3}$$

where P denotes the projection operator from the space of all vector fields $\mathcal{X}(M)$ to $\mathcal{X}_0(M)$ and Einstein's contraction rule is used for the repeated indices.

After some calculations, the equation of geodesics,

$$\tilde{\nabla}_X X = 0, \quad X = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t), \tag{4}$$

is shown to be written as

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \partial_i(B_{ik}B_{kj}) + \nabla p_\tau = 0, \tag{5}$$

$$\frac{\partial B_{ij}}{\partial t} + \partial_j(u_k B_{ik}) - \partial_i(u_k B_{jk}) = 0, \tag{6}$$

where p_τ is a scalar function which can be interpreted as the total pressure. Here we have introduced a generalized magnetic field $B = B_{ij}$ and a generalized vector potential A for the generalized current α by

$$A = -\Delta^{-1}\alpha, \quad B_{ij} = \partial_i A_j - \partial_j A_i. \tag{7}$$

These equations, together with the condition $\nabla \cdot u = 0$, are the equations of motion for an ideal MHD flow in N dimensions [13].

For $N = 3$, the Levi-Civita connection reduces to

$$\begin{aligned}
&\tilde{\nabla}_{(u,\alpha)^R}(v,\beta)^R|_{(h,\gamma)} \\
&= \tilde{R}_{(h,\gamma)} \Big(\ P[(u\cdot\nabla)v - \frac{1}{2}(\alpha \times B_\beta + \beta \times B_\alpha)], \\
&\quad \Big[\frac{1}{2}\nabla \times (-u \times \beta + v \times \alpha) \\
&\quad - \frac{1}{2}\nabla \times (\nabla \times (u \times B_\beta + v \times B_\alpha)) \Big] \Big),
\end{aligned} \tag{8}$$

where $B_\alpha = (B_{23}, B_{31}, B_{12})$ is a magnetic field on M satisfying

$$\nabla \times B_\alpha = \alpha, \quad \nabla \cdot B_\alpha = 0. \tag{9}$$

In the rest of the paper we consider the three-dimensional case.

3. Sectional curvatures

The sectional curvature of the section spanned by tangent vectors X, Y is defined as

$$K(X,Y) = \frac{R}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}, \tag{10}$$

where

$$R = \langle R(X,Y)Y, X\rangle, \tag{11}$$

$$R(X,Y) = \tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_Y\tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}. \tag{12}$$

We call R the tensor form of sectional curvature. Note that the sign of R coincides with that of $K(X,Y)$. The tensor form R is decomposed as

$$\begin{aligned}
R &= \langle R(u,v)v,u\rangle + 2\langle R(u,v)\beta,\alpha\rangle \\
&\quad + 2\langle R(u,\beta)v,\alpha\rangle + \langle R(u,\beta)\beta,u\rangle \\
&\quad + \langle R(\alpha,v)v,\alpha\rangle + \langle R(\alpha,\beta)\beta,\alpha\rangle
\end{aligned} \tag{13}$$

for $X = (u,\alpha)$, $Y = (v,\beta)$. Here we abbreviate $\langle R((u,0),(v,0))(0,\beta),(0,\alpha)\rangle$ as $\langle R(u,v)\beta,\alpha\rangle$ and so on. We need to calculate the sectional curvatures only at the identity $(e,0)$, since the right-invariance of the metric implies

$$K(X,Y) = K(\tilde{R}_{(g,\gamma)}X, \tilde{R}_{(g,\gamma)}Y). \tag{14}$$

We focus our attention on three typical sections here: (i) the pure hydrodynamic section; (ii) the pure magnetic section; and (iii) the section spanned by pure hydrodynamic vector and pure magnetic vector. The explicit expressions of two components $\langle R(u,v)\beta,\alpha\rangle$ and $\langle R(u,\beta)v,\alpha\rangle$ in (13) are given in the Appendix to complete the explicit expression of R for a general section.

3.1. Pure hydrodynamic section

Using the connection (8), we obtain the sectional curvature of the pure hydrodynamic section spanned by $(u,0),(v,0)$ as

$$\begin{aligned}
R_H &= \langle R(u,v)v,u\rangle \\
&= \langle Q[(v\cdot\nabla)v], Q[(u\cdot\nabla)u]\rangle \\
&\quad - |Q[(u\cdot\nabla)v]|^2
\end{aligned} \tag{15}$$

in tensor form. Here we denote by $Q = I - P$ the projection operator from $\mathcal{X}(M)$ to the space of all vector fields in gradient form $\mathcal{X}_0^{\perp}(M)$. Of course, the above expression (15) coincides with the expression of the sectional curvature for ideal HD [11]. It immediately follows that

Proposition 1 ([11])**.** If u satisfies $Q[(u \cdot \nabla)u] = 0$ (e.g., $u = (u_1(z), u_2(z), 0)$), then $R_H \leq 0$.

The condition $Q[(u \cdot \nabla)u] = 0$ implies that the pressure associated with u is zero, or in other words, u is pressureless. Thus any hydrodynamic section spanned by a pressureless field and arbitrary field has negative curvature.

3.2. Pure magnetic section

The sectional curvature of the pure magnetic section spanned by $(0, \alpha), (0, \beta)$ is calculated to be

$$\begin{aligned} R_M &= \langle R(\alpha, \beta)\beta, \alpha \rangle \\ &= -\langle P[(B_\beta \cdot \nabla)B_\beta], P[(B_\alpha \cdot \nabla)B_\alpha] \rangle \\ &\quad + \frac{1}{4}|P[(B_\alpha \cdot \nabla)B_\beta + (B_\beta \cdot \nabla)B_\alpha]|^2 \end{aligned} \tag{16}$$

in tensor form. From this expression we obtain

Proposition 2. If B_α satisfies $P[(B_\alpha \cdot \nabla)B_\alpha] = 0$, then $R_M \geq 0$.

The condition $P[(B_\alpha \cdot \nabla)B_\alpha] = 0$ implies that B_α is a "steady velocity field" in ideal HD, or equivalently, $u = B_\alpha$ is a steady solution of the Euler equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p. \tag{17}$$

For example, this condition is satisfied for the field that has Beltrami property $\nabla \times B_\alpha \parallel B_\alpha$; it is called a force-free field in MHD, since the Lorentz force $\alpha \times B_\alpha$ vanishes. An example of the proposition above is obtained by Zeitlin and Kambe [19].

3.3. Section spanned by pure hydrodynamic vector and pure magnetic vector

For the section spanned by $(u, 0)$ and $(0, \beta)$, the sectional curvature is calculated to be

$$\begin{aligned} R_{HM} &= \langle R(u, \beta)\beta, u \rangle \\ &= \langle P[(B_\beta \cdot \nabla)B_\beta], P[(u \cdot \nabla)u] \rangle \\ &\quad + \frac{1}{4}|\nabla \times (u \times B_\beta) + P[u \times \beta]|^2 \\ &\quad - |P[u \times \beta]|^2 \end{aligned} \tag{18}$$

in tensor form. We have

Proposition 3. If u or B_β is a "steady velocity field" and u is parallel to β everywhere, then $R_{HM} \geq 0$.

If u or B_β is a "steady velocity field" and u is parallel to B_β everywhere, then $R_{HM} \leq 0$.

As mentioned in the introduction, the sign of the sectional curvature is related with the stability of the geodesic flow on the semidirect group G. Unfortunately, there is no theorem connecting the stability of geodesic flow and the stability of velocity fields and current fields as far as the author knows. However, it seems to be true that when the velocity or current fields are unstable the corresponding geodesic flow is also unstable. In the following we show an example supporting this conjecture. The example corresponds to the well-known plasma instability, *sausage instability* [5].

Applying an infinitesimal velocity field ϵu to a steady current field α is interpreted as deformation of the current field α if we remind that the magnetic lines move with fluid particles, or in other words, they are in 'frozen-in state.'

Example 1 (sausage instability)**.** Let us consider the case where $\alpha = (0, 0, J)$ in the cylindrical coordinate system (r, θ, z) with

$$J = \begin{cases} \frac{I}{\pi a^2}, & \text{for } r < a \\ 0, & \text{for } r \geq a \end{cases} \tag{19}$$

$B = (0, B_\theta, 0)$ with

$$B_\theta = \begin{cases} \frac{Ir}{2\pi a^2}, & \text{for } r < a \\ \frac{I}{2\pi r}, & \text{for } r \geq a \end{cases} \tag{20}$$

and $u = (u_r(r, z), 0, u_z(r, z))$ with

$$u_r(r, z) = \begin{cases} \frac{f(z)r}{a^2}, & \text{for } r < a \\ \frac{f(z)}{r}, & \text{for } r \geq a \end{cases} \tag{21}$$

$$u_z(r, z) = \begin{cases} F(z), & \text{for } r < a \\ 0, & \text{for } r \geq a \end{cases} \tag{22}$$

where $F'(z) = -2f(z)/a^2$. The perturbation by the velocity field u is homogeneous in the poloidal or θ direction and it is regarded as a simple model of sausage instability. For simplicity, the function $f(z)$ is assumed to be periodic: $f(z + L) = f(z)$. The sectional curvature in this case is calculated to be

$$R_{HM} = -\frac{I^2}{4\pi a^4} \int_0^L \{f(z)\}^2 \, \mathrm{d}z. \tag{23}$$

Hence the section corresponding to the sausage instability has negative curvature.

The magnetic fields and the velocity field in Example 1 do not belong to C^1-class, but it is easy to show that slightly modified results for smoothed fields are obtained by using mollifier.

4. Summary

The sectional curvatures for ideal MHD were calculated for typical sections. Some results on the positiveness and negativeness of sectional curvatures were derived.

In particular, the sectional curvature for the section corresponding to the well-known plasma instability, sausage instability, was shown to be negative, in accordance with the physical phenomena. This result suggests that the present differential-geometric formulation is promising as a tool of practical analysis of MHD, though further study is expected.

Appendix

Here we show the expressions of $\langle R(u,v)\beta,\alpha\rangle$ and $\langle R(u,\beta)v,\alpha\rangle$ which appear in the tensor form R (13).

$$
\begin{aligned}
&\langle R(u,v)\beta,\alpha\rangle \\
&= -\frac{1}{4}\langle P[v\times\beta], P[u\times\alpha]\rangle - \frac{1}{4}\langle v\times\beta, \nabla\times(u\times B_\alpha)\rangle \\
&\quad - \frac{1}{4}\langle \nabla\times(v\times B_\beta), u\times\alpha\rangle \\
&\quad - \frac{1}{4}\langle \nabla\times(v\times B_\beta), \nabla\times(u\times B_\alpha)\rangle \\
&\quad + \frac{1}{4}\langle P[u\times\beta], P[v\times\alpha]\rangle + \frac{1}{4}\langle u\times\beta, \nabla\times(v\times B_\alpha)\rangle \\
&\quad + \frac{1}{4}\langle \nabla\times(u\times B_\beta), v\times\alpha\rangle \\
&\quad + \frac{1}{4}\langle \nabla\times(u\times B_\beta), \nabla\times(v\times B_\alpha)\rangle \\
&\quad + \frac{1}{2}\langle P[(u\cdot\nabla)v], \beta\times B_\alpha\rangle - \frac{1}{2}\langle P[(u\cdot\nabla)v], \alpha\times B_\beta\rangle \\
&\quad - \frac{1}{2}\langle P[(v\cdot\nabla)u], \beta\times B_\alpha\rangle + \frac{1}{2}\langle P[(v\cdot\nabla)u], \alpha\times B_\beta\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\langle R(u,\beta)v,\alpha\rangle \\
&= \frac{1}{4}\langle P[u\times\alpha], P[v\times\beta]\rangle + \frac{1}{2}\langle P[u\times\beta], P[v\times\alpha]\rangle \\
&\quad - \frac{1}{4}\langle u\times\beta, \nabla\times(v\times B_\alpha)\rangle - \frac{1}{4}\langle v\times\alpha, \nabla\times(u\times B_\beta)\rangle \\
&\quad - \frac{1}{4}\langle \nabla\times(u\times B_\alpha), \nabla\times(v\times B_\beta)\rangle \\
&\quad - \frac{1}{4}\langle P[(u\cdot\nabla)v], \alpha\times B_\beta\rangle - \frac{1}{4}\langle P[(u\cdot\nabla)v], \beta\times B_\alpha\rangle \\
&\quad - \frac{1}{4}\langle P[(v\cdot\nabla)u], \alpha\times B_\beta\rangle - \frac{1}{4}\langle P[(v\cdot\nabla)u], \beta\times B_\alpha\rangle.
\end{aligned}
$$

The formula $\langle \tilde{\nabla}_X\tilde{\nabla}_Y Z, W\rangle = -\langle \tilde{\nabla}_Y Z, \tilde{\nabla}_X W\rangle$ which is valid for right-invariant fields X, Y, Z, W is useful in deriving these expressions. The sum of these two components reduces to be

$$
\begin{aligned}
&\langle R(u,v)\beta,\alpha\rangle + \langle R(u,\beta)v,\alpha\rangle \\
&= -\frac{1}{4}\langle v\times\beta, \nabla\times(u\times B_\alpha)\rangle - \frac{1}{4}\langle \nabla\times(v\times B_\beta), u\times\alpha\rangle \\
&\quad - \frac{1}{4}\langle \nabla\times(v\times B_\beta), \nabla\times(u\times B_\alpha)\rangle \\
&\quad + \frac{3}{4}\langle P[u\times\beta], P[v\times\alpha]\rangle + \frac{1}{4}\langle P[(u\cdot\nabla)v], \beta\times B_\alpha\rangle \\
&\quad - \frac{3}{4}\langle P[(u\cdot\nabla)v], \alpha\times B_\beta\rangle - \frac{3}{4}\langle P[(v\cdot\nabla)u], \beta\times B_\alpha\rangle \\
&\quad + \frac{1}{4}\langle P[(v\cdot\nabla)u], \alpha\times B_\beta\rangle.
\end{aligned}
$$

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