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# Complete coset weight distributions of second order Reed-Muller code of length 64

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**Abstract.** In this paper we determine the complete coset weight distributions of the second order Reed-Muller code $RM(2,6)$ of length 64. Our method fully uses the interaction between the Jacobi polynomials for the code $RM(2,6)$ and those of the dual code $RM(3,6)$. The method also employs the group theoretic reduction processes to diminish the runtimes of computing the Jacobi polynomials for the code $RM(2,6)$ in great effect.

*Keywords.* second order Reed-Muller code of length 64, coset weight distributions, Jacobi polynomials

## 1. Introduction

### 1.1. An Overview of the Problem and the Way to attack the Problem

Reed-Muller codes $RM(r,m)$ are a class of binary linear codes of length $2^m$ and of dimension $1+\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{r}$ $(r \le m)$. For precise definitions concerning Reed-Muller codes one may refer the book by MacWilliams-Sloane [7], Chapters 13-15.

As to the first order Reed-Muller code $RM(1,6)$ of length 64 the covering radius was determined by T.Helleseth ,T. Kl∅ve, and J. Mykkeltveit [6]. The complete coset weight distribution of $RM(1,6)$ is described in Maiorana [8].

The problem of determining the covering radius of the second order Reed-Muller code $RM(2,6)$ of length 64 is settled by J. Schatz [18] and its value is 18. In the present paper we will determine the complete coset weight distributions of $RM(2,6)$ together with their frequencies.

A naive approach to our present problem may involve the enormity of time consumption. If we use simple nested loops of depth up to 18 in the programs of computing cosets, it may take more than hundred years. Therefore we must devise some mathematical ways to avoid this difficulty. In the Section 6 we will develop two group theoretic techniques that largely reduce the runtime of computing cosets. Without these techniques our present project would not terminate within a reasonable period.

Some words for our methodology. If the binary code is self-dual, then the invariant theory plays an important role in determining complete coset weight distributions of the code, since the Jacobi polynomials associated with the self-dual code are simultaneously invariant polynomials for a certain finite unitary reflection group (as references one may see [11],[12],[13],[14]). When the code is not self-dual, then the invariant theory does not work. In place of using invariant theory we use the interplay between the Jacobi polynomials for the code $C$ and those for the dual code $C^{\perp}$. This idea works well if the code is self-orthogonal, in particular when $C = RM(2,6)$ and $C^{\perp} = RM(3,6)$. Jacobi polynomials play multiple roles: (i) first through them we can determine the complete weight distributions together with their multiplicities for the code $RM(3,6)$, (ii) second the frequencies of the coset weight distributions of $RM(2,6)$ of larger weights are controlled by those of $RM(3,6)$ through the Jacobi polynomials of both codes $RM(3,6)$ and $RM(2,6)$. After computing all the coset weight distributions of $RM(2,6)$ we want to verify that our computations are correct ones. This is done by three ways. The first is of elemetary combinatorial nature (Conf. Section 7.1), the second is to use Delsarte's identities (Conf. Section 7.2), and the third is to use the partial coincidence properties between the coset weight distributions of $RM(3,6)$ and $RM(2,6)$ (Conf. Section 7.3). The last way sometimes helps us to find hidden coset weight distributions of $RM(2,6)$ that are not easily detectable at first sight.

One novel feature of our result is that there are several cases that even if the coset weight enumerators of two different cosets of the same coset weight in the code $RM(2,6)$ have identical coset weight enumerators, when they are considered in the dual code $RM(3,6)$, may be embedded into the different cosets with different coset weight enumerators. It is provable algebraically that the pair $RM(1,5)$ and $RM(3,5)$ and the pair $RM(1,6)$ and $RM(4,6)$ does not have this phenomenon. This happens in the cosets of weights 10, 12 and 14 respectively. We may call them optical isomeric cosets.

We expect that our group theoretical process to reduce the runtime of the computation and the method to treat the coset weight distributions of the code $C$ and $C^{\perp}$ simultaneously would have a wider significance for further investigations in the problems of this direction.

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### 1.2. Standard Definitions from Binary Codes

Let $\mathbf{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbf{F}_2^n$ be the vector space of dimension $n$ over $\mathbf{F}_2$ . A linear $[n,k]$ code $\mathbf{C}$ is a vector subspace of $V$ of dimension $k$. An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. In $V$ , the inner product , which is denoted by $(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y}$ in $V$, is defined as usual. Two codes $\mathbf{C}_1$ and $\mathbf{C}_2$ are said to be equivalent if and only if after a permutation of coordinate positions of $\mathbf{C}_1$ we get $\mathbf{C}_2$. An automorphism of the code $\mathbf{C}$ is a permutation of the coordinates sending $\mathbf{C}$ onto itself. All automorphisms of $\mathbf{C}$ forms a group, and it is denoted by $Aut(\mathbf{C})$. The dual code $\mathbf{C}^{\perp}$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^{\perp} = \{\mathbf{u} \in V \mid (\mathbf{u},\mathbf{v}) = \mathbf{0} \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^{\perp}$, and the code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^{\perp}$.

Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

be a vector in $V$ , then the Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of $i$'s such that $x_i \neq 0$. The Hamming distance $d$ on $V$ is defined by $d(\mathbf{x},\mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$. Let $\mathbf{C}$ be a code ,then the minimum distance $d$ of the code $\mathbf{C}$ is defined by

$$\begin{aligned} d &= \mathrm{Min}_{\mathbf{x},\mathbf{y}\in\mathbf{C},\mathbf{x}\neq\mathbf{y}} d(\mathbf{x},\mathbf{y}) \\ &= \mathrm{Min}_{\mathbf{x}\in\mathbf{C},\mathbf{x}\neq\mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

A vector $\mathbf{v}$ in a coset $U$ in $V/\mathbf{C}$ is a coset leader if $\mathbf{v}$ satisfies

$$wt(\mathbf{v}) \leq wt(\mathbf{z}) \text{ for all } \mathbf{z} \in U.$$

The weight of a coset $U$ is the weight of the coset leader of $U$.

The inhomogeneous weight enumerator $W_{\mathbf{C}}(X)$ of a code $\mathbf{C}$ is defined by

$$\begin{aligned} W_{\mathbf{C}}(X) &= \sum_{\mathbf{v}\in\mathbf{C}} X^{wt(\mathbf{v})} \\ &= \sum_{r=0}^{n} a_r X^r, \end{aligned}$$

where $a_r$ is the number of the codewords $\mathbf{v}$ of weight $r$ in $\mathbf{C}$ The homogeneous weight enumerator of the code $\mathbf{C}$ is defined by :

$$W_{\mathbf{C}}(x,y) = \sum_{r=0}^{n} a_r x^{n-r} y^r.$$

A celebrated MacWilliams identity says :

**Theorem 1.** *Let $W_{\mathbf{C}}(X)$ be the weight enumerator of a binary code, then the following identity holds :*

$$W_{\mathbf{C}^{\perp}}(X) = \frac{1}{\mid \mathbf{C} \mid}(1+X)^n W_{\mathbf{C}}(\frac{1-X}{1+X}).$$

The homogeneous version of the above theorem is

**Theorem 2.**

$$W_{\mathbf{C}^{\perp}}(x,y) = \frac{1}{\mid \mathbf{C} \mid} W_{\mathbf{C}}(x+y, x-y)$$

The coset weight enumerator of a coset $U$ in a code $\mathbf{C}$ is defined to be a polynomial in $X$ given by

$$W_U(X) = \sum_{\mathbf{z}\in U} X^{wt(\mathbf{z})}.$$

### 1.3. Statement of the Problem

Let $\mathbf{x}$ be a vector in $V$, then the Hamming sphere $S_t(\mathbf{x})$ of radius $t$ with center $\mathbf{x}$ is defined by

$$S_t(\mathbf{x}) = \{\mathbf{y} \in V \mid d(\mathbf{y},\mathbf{x}) \leq t\}.$$

The covering radius $t(\mathbf{C})$ of the code $\mathbf{C}$ is defined to be the smallest integer $t$ such that the equation

$$V = \bigcup_{\mathbf{x}\in\mathbf{C}} S_t(\mathbf{x})$$

holds.

The covering radius of a code $\mathbf{C}$ is also defined by another way.

**Proposition 1.** *It holds that*

$$t(\mathbf{C}) = \max_{\mathbf{u}\in\mathbf{F}_2^n}(\min_{\mathbf{z}\in\mathbf{u}+\mathbf{C}} wt(\mathbf{z})).$$

So to say the covering radius of $\mathbf{C}$ is the weight of a coset leader of greatest weight.

A more precise problem is that for a given code $\mathbf{C}$ determine all the coset weight enumerators ( or equally the coset weight distributions ) $W_U(X)$ together with their frequencies.

In the present paper we will determine all the coset weight enumerators with their frequencies for the second order Reed-Muller code $RM(2,6)$ of length 64.

## 2. Our Method of attacking the Problem

The process goes as follows. First we determine the coefficients of the coset weight enumerators of weights up to 8 by way of computing Jacobi polynomials for the code $RM(2,6)$ which is introduced in Ozeki [11] and fully used in the self-dual code cases [12],[13]. As a byproduct we can determine the coset weight enumerators of the cosets in the dual code $RM(3,6)$ through the MacWilliams type transformation between the Jacobi polynomials of $RM(2,6)$ and

those of $RM(3,6)$.
For instance let

$$\begin{aligned}
&Jac(RM(2,6),\mathbf{v} \mid X,Z) = \\
&\quad 1 + (a_8Z^8 + a_7Z^7 + a_6Z^6 + a_5Z^5 + a_4Z^4 + a_3Z^3 \\
&\quad +a_2Z^2 + a_1Z + a_0)X^{16} \\
&\quad +(b_8Z^8 + b_7Z^7 + b_6Z^6 + b_5Z^5 + b_4Z^4 + b_3Z^3 \\
&\quad +b_2Z^2 + b_1Z + b_0)X^{24} \\
&\quad +(c_8Z^8 + c_7Z^7 + c_6Z^6 + c_5Z^5 + c_4Z^4 \\
&\quad +c_3Z^3 + c_2Z^2 + c_1Z + c_0)X^{28} \\
&\quad +(d_8Z^8 + d_7Z^7 + d_6Z^6 + d_5Z^5 + d_4Z^4 \\
&\quad +d_5Z^3 + d_6Z^2 + d_7Z + d_8)X^{32} + \cdots
\end{aligned}$$

be the Jacobi polynomial of $RM(2,6)$ with a reference vector $\mathbf{v}$ of weight 8, then the coset weight enumerator of the coset $U = \mathbf{v} + RM(2,6)$ is given by

$$\begin{aligned}
&W_U(X) = \\
&\quad (a_8+1)X^8 + a_7X^{10} + a_6X^{12} + a_5X^{14} + (a_4+b_8)X^{16} \\
&\quad +(a_3+b_7)X^{18} + (a_2+b_6+c_8)X^{20} + (a_1+b_5+c_7)X^{22} \\
&\quad +(a_0+b_4+c_6+d_8)X^{24} + (b_3+c_5+d_7)X^{26} \\
&\quad +(b_2+c_4+c_0+d_6)X^{28} + (b_1+c_3+c_1+d_5)X^{30} \\
&\quad +(2b_0+2c_2+d_4)X^{32} + \cdots .
\end{aligned}$$

One may note that the coset weight enumerators and Jacobi polynomials for the binary linear codes having all one vector, such as RM(2,6) or RM(3,6), have palindromic coefficients. To get an expression for the coset weight enumerator of the coset $\mathbf{v} + RM(3,6)$ we put

$$\mu_1 = \sum_{i=1}^{8}(-1)^i a_i,\ \mu_2 = \sum_{i=1}^{8}(-1)^i b_i,$$
$$\mu_3 = \sum_{i=1}^{8}(-1)^i c_i, \mu_4 = \sum_{i=1}^{8}(-1)^i d_i,$$

then we get

$$W_{\mathbf{v}+RM(3,6)}(X) = \tilde{a}_0 + \tilde{a}_2X^2 + \tilde{a}_4X^4 + \tilde{a}_6X^6 + \tilde{a}_8X^8 + \cdots ,$$

where $\tilde{a}_0 = (2\mu_1 + 2\mu_2 + 2\mu_3 + \mu_4 + 2)/4194304$,
$\tilde{a}_2 = (960\mu_1 + 192\mu_2 - 32\mu_4 + 4032)/4194304$,
$\tilde{a}_4 = (56288\mu_1 - 1568\mu_2 - 672\mu_3 + 496\mu_4 + 1270752)/4194304$,
$\tilde{a}_6 = (779584\mu_1 - 9152\mu_2 + 10752\mu_3 - 4960\mu_4 + 149948736)/4194304$,
$\tilde{a}_8 = (116976\mu_1 + 181504\mu_2 - 91152\mu_3 + 35960\mu_4 + 8852330736)/4194304, \cdots$. Same kind of formulas are obtained in other coset weight cases.
These formulas can be implemented into the programmings.

In Section 6 we have made two group theoretic techniques to reduce the runtimes of our main programmings. Without those we could not finish our computations within a reasonable period.

As checks for the correctness of our computation we devised three methods. First is to compare the sum of all the coset weight enumerators with its multiplicities with the binomial coefficients. Second one is to use the so called Delsarte's relation in the theory of Hamming association scheme. Third one is newly devised. All these methods are more precisely explained in Section 7.

## 3. Algebraic Tools

### 3.1. Jacobi polynomials for Binary codes.

Let $\mathbf{u} * \mathbf{v}$ be the number of common 1's in the entries of the vectors $\mathbf{u}, \mathbf{v} \in V$, then Jacobi polynomial $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ for $\mathbf{C}$ with respect to a reference vector $\mathbf{v} \in \mathbf{F}_2^n$ is defined by

$$Jac(\mathbf{C}, \mathbf{v} \mid X, Z) = \sum_{\mathbf{u} \in \mathbf{C}} X^{\mathbf{u}*\mathbf{u}} Z^{\mathbf{u}*\mathbf{v}}.$$

The homogeneous Jacobi polynomial is defined by

$$Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) = \sum_{\mathbf{u} \in \mathbf{C}} x^{n - wt(\mathbf{v}) - wt(\mathbf{u}) + \mathbf{u}*\mathbf{v}} y^{wt(\mathbf{u}) - \mathbf{u}*\mathbf{v}} u^{wt(\mathbf{v}) - \mathbf{u}*\mathbf{v}} v^{\mathbf{u}*\mathbf{v}},$$

, where $x, y, u, v$ are algebraically independent variables over $\mathbf{C}$. The connection between inhomogeneous Jacobi and homogeneous Jacobi is described by

$$Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) \ = \ x^{n-wt(\mathbf{v})} u^{wt(\mathbf{v})} Jac_i(\mathbf{C}, \mathbf{v}, yx, xvyu),$$

and

$$Jac_i(\mathbf{C}, \mathbf{v}, X, Z) = Jac(\mathbf{C}, \mathbf{v}, 1, X, 1, XZ).$$

One of our basic theorems is :

**Theorem 3.** *Let $\mathbf{C}$ be a binary code of length $n$ and $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ a Jacobi polynomial for the code $\mathbf{C}$ with any binary vector $\mathbf{v} \in \mathbf{F}_2^n$, then it holds that*

$$\begin{aligned}
&Jac(\mathbf{C}^\perp, \mathbf{v} \mid X, Z) \\
&= \frac{1}{|\mathbf{C}|}(1+X)^n \left(\frac{1+XZ}{1+X}\right)^{wt(\mathbf{v})} \\
&\quad \times Jac(\mathbf{C}, v \mid \frac{1-X}{1+X}, \frac{(1-XZ)(1+X)}{(1+XZ)(1-X)}).
\end{aligned}$$

There is a homogeneous version of the above theorem :

**Theorem 4.** *Let $\mathbf{C}$ be a binary linear code of length $n$, then $Jac(\mathbf{C}, \mathbf{v}, x, y, u, v)$ satisfies*

(1)
$$Jac(\mathbf{C}^\perp, \mathbf{v}, x, y, u, v) = \frac{1}{|\mathbf{C}|} Jac(\mathbf{C}, \mathbf{v}, x+y, x-y, u+v, u-v)$$

### 3.2. The numbers $p_{i,j}^{(k)}$

We need the numbers $p_{i,j}^{(k)}$ that are connected with the Krawtchouk polynomials $P_k(x)$ (see [5] for the reference). Here the numbers $p_{i,j}^{(k)}$ are integers determined as follows. We put

$$D_i = \{\mathbf{a} \in \mathbf{F}_2^n \mid wt(\mathbf{a}) = i\},$$

then first for a fixed $\mathbf{c} \in D_k$ the number of pairs $\mathbf{a}$ and $\mathbf{b}$ :

$$p_{i,j}^{(k)} = \#\{<\mathbf{a},\mathbf{b}> \mid \mathbf{a} \in D_i,\ \mathbf{b} \in D_j,\ \mathbf{a}+\mathbf{b} = \mathbf{c}\ \}$$

is proved to be independent of the choice of $\mathbf{c}$. As to the formula for $p_{i,j}^{(k)}$ one may refer MacWilliams and Sloane [7], Ch. 21 section 3. The formula reads

(2)

$$p_{i,j}^{(k)} = \begin{cases} \binom{k}{\frac{i-j+k}{2}}\binom{n-k}{\frac{i+j-k}{2}} & \text{if } i+j \equiv k \pmod 2 \\ 0 & \text{if } i+j \equiv k+1 \pmod 2 \end{cases}$$

### 3.3. Distance Matrix

Let $\mathbf{C}$ be a binary linear [n,k] code. The $2^n \times (n+1)$ matrix $\mathbf{B} = (B_i(\mathbf{e}))_{\mathbf{e}\in\mathbf{F}_2^n, 0\le i\le n}$, is called ,after Delsarte [5], the distance matrix for the code $\mathbf{C}$. Here the entries of $\mathbf{B}$ are defined by

$$B_i(\mathbf{e}) = \#\{\ \mathbf{a} \in \mathbf{C} \mid d(\mathbf{a},\mathbf{e}) = i\ \} \quad 0 \le i \le n.$$

More concretely we give

$$\mathbf{B} = \begin{pmatrix} B_0(\mathbf{e}_1) & B_1(\mathbf{e}_1) & B_2(\mathbf{e}_1) & \cdots & B_n(\mathbf{e}_1) \\ B_0(\mathbf{e}_2) & B_1(\mathbf{e}_2) & B_2(\mathbf{e}_2) & \cdots & B_n(\mathbf{e}_2) \\ \vdots & & & & \\ B_0(\mathbf{e}_m) & B_1(\mathbf{e}_m) & B_2(\mathbf{e}_m) & \cdots & B_n(\mathbf{e}_m) \\ \vdots & & & & \\ B_0(\mathbf{e}_{2^n}) & B_1(\mathbf{e}_{2^n}) & B_2(\mathbf{e}_{2^n}) & \cdots & B_n(\mathbf{e}_{2^n}) \end{pmatrix}.$$

A formula in [5] is very useful for our present work. It is

$$\text{(3)} \qquad \sum_{\mathbf{e}\in\mathbf{F}_2^n} B_i(\mathbf{e})B_j(\mathbf{e}) = M\sum_{k=0}^{n} p_{i,j}^{(k)} A_k(\mathbf{C}).$$

Here $M = 2^r$, $r = dim\mathbf{C}$, and $A_k(\mathbf{C})$ is the number of codewords of weight $k$ in the code $\mathbf{C}$. The above formula gives us vertical informations on $\mathbf{B}$.
One may remark that

$$\begin{aligned} B_i(\mathbf{e}) &= \#\{\ \mathbf{a} \in \mathbf{C} \mid d(\mathbf{a},\mathbf{e}) = i\} \\ &= \#\{\ \mathbf{a} \in \mathbf{C} \mid wt(\mathbf{a}+\mathbf{e}) = i\} \\ &= \#\{\ \mathbf{b} \in \mathbf{e}+\mathbf{C} \mid wt(\mathbf{b}) = i\ \}, \end{aligned}$$

so that

$$B_0(\mathbf{e}), B_1(\mathbf{e}), \cdots, B_n(\mathbf{e})$$

is the coset weight distribution of the coset $\mathbf{e}+\mathbf{C}$. In other words the polynomial

$$\sum_{i=0}^{n} B_i(\mathbf{e})X^i$$

is the inhomogeneous coset weight enumerator of the coset $\mathbf{e}+\mathbf{C}$, and the polynomial

$$\sum_{i=0}^{n} B_i(\mathbf{e})x^{n-i}y^i$$

is the homogeneous one.

### 3.4. A Modification of Distance Matrix

We remark that if two vectors $\mathbf{e}_1$ is congruent to $\mathbf{e}_2$ modulo $\mathbf{C}$ then it holds that $B_i(\mathbf{e}_1) = B_i(\mathbf{e}_2)$ for $0 \le i \le n$, and so that we may rewrite the formula (3) into

$$\text{(4)} \qquad \sum_{\mathbf{e}\in\mathbf{F}_2^n/\mathbf{C}} B_i(\mathbf{e})B_j(\mathbf{e}) = \sum_{k=0}^{n} p_{i,j}^{(k)} A_k(\mathbf{C}).$$

Accordingly we may use the shortened distance matrix $\mathbf{B}_S$ :

$$\mathbf{B}_S = (B_i(\mathbf{e}))_{\mathbf{e}\in\mathbf{F}_2^n/\mathbf{C}, 0\le i\le n}$$

or more concretely

$$\mathbf{B}_S = \begin{pmatrix} B_0(\mathbf{e}_1) & B_1(\mathbf{e}_1) & B_2(\mathbf{e}_1) & \cdots & B_n(\mathbf{e}_1) \\ B_0(\mathbf{e}_2) & B_1(\mathbf{e}_2) & B_2(\mathbf{e}_2) & \cdots & B_n(\mathbf{e}_2) \\ \vdots & & & & \\ B_0(\mathbf{e}_m) & B_1(\mathbf{e}_m) & B_2(\mathbf{e}_m) & \cdots & B_n(\mathbf{e}_m) \\ \vdots & & & & \\ B_0(\mathbf{e}_t) & B_1(\mathbf{e}_t) & B_2(\mathbf{e}_t) & \cdots & B_n(\mathbf{e}_t) \end{pmatrix},$$

with $t = 2^{n-r}, r = \dim\mathbf{C}$.

## 4. Relation between Jacobi polynomials of codes and coset weight enumerator

We quote two theorems from [12], which are originally valid for self-dual code but are also valid for self-orthogonal codes:

**Theorem 5.** *Let $\mathbf{C}$ be a binary linear code of length $n$. We take any vector $\mathbf{v}$ in $\mathbf{F}_2^n$. If $\mathbf{v}+\mathbf{C}$ is the coset in $\mathbf{F}_2^n/\mathbf{C}$ to which $\mathbf{v}$ belongs, then we have*

$$\begin{aligned} & W_{\mathbf{v}+\mathbf{C}}(X) \\ &= \psi(Jac(C,\mathbf{v} \mid X,Z)) \\ &= X^{wt(\mathbf{v})}Jac(C,\mathbf{v} \mid X,X^{-2}), \end{aligned}$$

*where $Jac(C,\mathbf{v} \mid X,Z)$ is the inhomogeneous Jacobi polynomial for the code $\mathbf{C}$ with respect to $\mathbf{v}$.*

The reference vector $\mathbf{v}$ of $Jac(C, \mathbf{v} \mid X, Z)$ is called rigid if $\mathbf{v}$ is a coset leader of the coset $\mathbf{v} + \mathbf{C}$. Let $\mathbf{R}_k(J_0)$ be the set of rigid vectors in $\mathbf{C}$ of a fixed weight $k$ with equal rigid Jacobi polynomials $J_0 = Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$.

**Theorem 6.** *Let $\mathbf{C}$ be a linear binary code of length $n$. Suppose that in a binary linear code $\mathbf{C}$ any two different rigid Jacobi polynomials derive (via the process in Theorem 5) different coset weight enumerators. Let*

$$\begin{aligned} X^k Jac(\mathbf{C}, \mathbf{v} \mid X, X^{-2}) &= W_{\mathbf{v}+\mathbf{C}}(X) \\ &= a_k X^k + a_{k+1} X^{k+1} + \cdots \end{aligned} \tag{5}$$

*be its common coset weight enumerator for the vector in $\mathbf{R}_k(J_0)$, then the number of the different cosets $\mathbf{S}$ in $\mathbf{F}_2^n/\mathbf{C}$ such that*

$$W_{\mathbf{S}}(X) = W_{\mathbf{v}+\mathbf{C}}(X)$$

*is given by* $\dfrac{\mid \mathbf{R}_k(J_0) \mid}{a_k}$.

Jacobi polynomials together with this theorem give us clearer insights for the horizontal information on the matrix $\mathbf{B}$ or $\mathbf{B}_S$.

$$\begin{array}{ccc} & \text{MacWilliams} & \\ & \text{transform} & \\ Jac(C, \mathbf{v} \mid X, Z) & \longleftrightarrow & Jac(C^{\perp}, \mathbf{v} \mid X, Z) \\ \downarrow \psi & & \downarrow \psi \\ W_{\mathbf{v}+\mathbf{C}}(X) & < -- > & W_{\mathbf{v}+\mathbf{C}^{\perp}}(X) \end{array}$$

One may note that there is no direct algebraic correspondence such as MacWilliams transform between $W_{\mathbf{v}+\mathbf{C}}(X)$ and $W_{\mathbf{v}+\mathbf{C}^{\perp}}(X)$.

An easy criterion of the rigidity of the vector $\mathbf{v}$ says

**Proposition 2.** *A necessary and sufficient condition for the rigidity of the vector $\mathbf{v}$ is that the vector $\mathbf{v}$ satisfies the following inequalities*

$$\mathbf{u} * \mathbf{v} \leq 1/2 wt(\mathbf{u})$$

*holds for any $\mathbf{u} \in \mathbf{C}$.*

We may skip the proof of this proposition.

## 5. Preliminary result

When we ask the computer algebra MAGMA in such a way that

```
> R2:=ReedMullerCode(3,6);
> CD:=CosetDistanceDistribution(R2);
> CD;
```

the answer of MAGMA is

```
[ <0, 1>, <1, 64>, <2, 2016>, <3, 41664>,
 <4, 313131>, <5, 1166592>, <6, 1768116>,
 <7, 888832>, <8, 13888> ]
```

The meaning of the above output is that in the third order Reed-Muller code of length $2^6 = 64$ the number of coset of weight 0 is 1, the number of the cosets of weight 1 is 64 and so on. But MAGMA (in our version) does not answer for the question such as

```
> R:=ReedMullerCode(2,6);
> CD:=CosetDistanceDistribution(R);
> CD;
```

It is relatively easy to enumerate all the coset weight enumerators of the cosets in $RM(3,6)$. This is performed by computing the Jacobi polynomials $Jac(RM(2,6), \mathbf{v} \mid X, Z)$ with $wt(\mathbf{v}) \leq 8$ together with Delsarte's relation described in Section 7. All cosets weight enumerators in $RM(3,6)$ are given here:

$$\begin{aligned} \Phi_0(X) = \; & 1 + 11160X^8 + 1749888X^{12} + 22855680X^{14} \\ & +232081500X^{16} + 1717223424X^{18} + 9366150528X^{20} \\ & +38269550592X^{22} + 119637587496X^{24} \\ & +286573658112X^{26} + 533982211840X^{28} \\ & +771854598144X^{30} + 874731154374X^{32} + \cdots \end{aligned}$$

$$\begin{aligned} \Phi_1(X) = \; & X + 1395X^7 + 9765X^9 + 328104X^{11} + 6421464X^{13} \\ & +75876375X^{15} + 657030213X^{17} + 4161176376X^{19} \\ & +19594386504X^{21} + 69978487887X^{23} \\ & +191194040793X^{25} + 403770327184X^{27} \\ & +662171837040X^{29} + 847413332451X^{31} + \cdots \end{aligned}$$

$$\begin{aligned} \Phi_2(X) = \; & X^2 + 155X^6 + 2480X^8 + 65813X^{10} + 1573312X^{12} \\ & +22901343X^{14} + 232624992X^{16} + 1717865837X^{18} \\ & +9355061952X^{20} + 38314195087X^{22} \\ & +119513197392X^{24} + 286853853441X^{26} \\ & +533468304256X^{28} + 772602746099X^{30} \\ & +873881726784X^{32} + \cdots \end{aligned}$$

$$\begin{aligned} \Phi_3(X) = \; & X^3 + 15X^5 + 420X^7 + 12540X^9 + 351249X^{11} \\ & +6283679X^{13} + 76053456X^{15} + 657637488X^{17} \\ & +4158076045X^{19} + 19602358787X^{21} \\ & +69960579924X^{23} + 191231290380X^{25} \\ & +403709972509X^{27} + 662235446163X^{29} \\ & +847385192896X^{31} + \cdots \end{aligned}$$

$$\begin{aligned} \Phi_{4,1}(X) = \; & 2X^4 + 56X^6 + 2100X^8 + 71576X^{10} + 1566454X^{12} \\ & +22829520X^{14} + 232926664X^{16} + 1717363792X^{18} \\ & +9355683090X^{20} + 38312345128X^{22} \\ & +119518499596X^{24} + 286846272264X^{26} \\ & +533470417910X^{28} + 772612745440X^{30} \\ & +873865063920X^{32} + \cdots \end{aligned}$$

$\Phi_{4,2}(X) =$
$$16X^4 + 3360X^8 + 61440X^{10} + 1596848X^{12} +22794240X^{14} + 232981824X^{16} + 1716981760X^{18} +9356785296X^{20} + 38311473152X^{22} +119516468448X^{24} + 286851686400X^{26} +533466934192X^{28} + 772608630784X^{30} +873873715584X^{32} + \cdots$$

$\Phi_5(X) =$
$$6X^5 + 298X^7 + 13090X^9 + 354510X^{11} + 6264276X^{13} +76069196X^{15} + 657713004X^{17} + 4157996212X^{19} +19601835330X^{21} + 69962104398X^{23} +191230227494X^{25} + 403708040490X^{27} +662239889976X^{29} + 847382747272X^{31} + \cdots$$

$\Phi_{6,1}(X) =$
$$32X^6 + 2112X^8 + 72352X^{10} + 1565952X^{12} +22817472X^{14} + 232951936X^{16} + 1717428416X^{18} +9355366656X^{20} + 38312701536X^{22} +119518868416X^{24} + 286844924640X^{26} +533471398400X^{28} + 772613683328X^{30} +873862948608X^{32} + \cdots$$

$\Phi_{6,2}(X) =$
$$64X^6 + 1920X^8 + 73024X^{10} + 1562112X^{12} +22834560X^{14} + 232918784X^{16} + 1717418368X^{18} +9355525632X^{20} + 38312469696X^{22} +119518789760X^{24} + 286845498816X^{26} +533470915584X^{28} + 772613333248X^{30} +873863827968X^{32} + \cdots$$

$\Phi_7(X) =$
$$288X^7 + 13216X^9 + 354816X^{11} + 6263040X^{13} +76059072X^{15} + 657768384X^{17} + 4157911296X^{19} +19601776128X^{21} + 69962467680X^{23} +191229859296X^{25} + 403707781120X^{27} +662240761344X^{29} + 847382239872X^{31} + \cdots$$

$\Phi_8(X) =$
$$2304X^8 + 71680X^{10} + 1569792X^{12} + 22800384X^{14} +232985088X^{16} + 1717438464X^{18} + 9355207680X^{20} +38312933376X^{22} + 119518947072X^{24} +286844350464X^{26} + 533471881216X^{28} +772614033408X^{30} + 873862069248X^{32} + \cdots$$

In the above enumeration we cut off the terms after the power $X^{32}$, since the coefficients go in a palindromic way with the center $X^{32}$. In later occasions this cutting off process will be applied whenever it works.

Table 1.
Table of the multiplicities of
the coset weight enumerators in $RM(3,6)$.

| coset weight enumerator | multiplicity |
|---|---|
| $\Phi_0$ | 1 |
| $\Phi_1$ | 64 |
| $\Phi_2$ | 2016 |
| $\Phi_3$ | 41664 |
| $\Phi_{4,1}$ | 312480 |
| $\Phi_{4,2}$ | 651 |
| $\Phi_5$ | 1166592 |
| $\Phi_{6,1}$ | 1749888 |
| $\Phi_{6,2}$ | 18228 |
| $\Phi_7$ | 888832 |
| $\Phi_8$ | 138888 |

The above table by far refines the answer of MAGMA for $RM(3,6)$.

When we want to know concerning the complete coset distance distribution of $RM(2,6)$, we must explore some algebraic techniques. This is our motivation of the following sections.

## 6. How to reduce the runtime

Our plan to determine the coset weight distributions of $RM(2,6)$ is first to compute the Jacobi polynomials $Jac(\mathbf{v}, X, Z)$ for each reference vectors $\mathbf{v}$ of weight $k$. Among these vectors we select rigid vectors, and from these vectors we get coset weight enumerators of coset weight $k$. A naive idea to determine the coset weight enumerators of coset weight $k$ is to use nested loops of depth $k$ in the programs. The run time increases greatly as $k$ grows . We give the estimates of runtime on naive programs below, some of which were tested in the actual programs.

Table 3. Run time estimate for naive programs.

| $k$ | run time (minutes) | days |
|---|---|---|
| $0 \leq k \leq 7$ | tolerable | within a day |
| 8 | 1840 | 1 day and 7 hours |
| 9 | 11448 | 8 days |
| 10 | 62968 | 43 days |
| 11 | 309120 | 214 days |
| 12 | 1365280 | 948 days |
| 13 | 5461120 | 3792 days |
| 14 | 19894080 | 13815 days |
| $15 \leq k \leq 18$ | awful runtimes | |

### 6.1. Group theoretic process I

To save the runtime we devised the following process.

We divide 64 coordinates into 16 blocks so that each block consists of 4 coordinates. The division is done in a natural order. Consider $G = Aut(RM(2,6))$ the group of automorphisms of the second order Reed-Muller code of length 64. $G$ acts on the totality of the blocks. The group $G$ is constructed by **GUAVA** [3] (GAP package for computing with error-correcting codes). The order of $G$ is $2^{21} \cdot 3^4 \cdot 5 \cdot 7 \cdot 31$. We find a subgroup $K$ of $G$ which preserves the set of blocks. The order of $K$ is $2^{21} \cdot 3^3 \cdot 5 \cdot 7$. We construct $\widetilde{K}$ as a subgroup of the symmetric group of degree 16 from the action of $K$ on 16 blocks. The order of $\widetilde{K}$ is $2^{10} \cdot 3^2 \cdot 5 \cdot 7$. We remark that two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, which are connected by $\mathbf{v}_2 = \sigma(\mathbf{v}_1)$ with $\sigma \in Aut(RM(2,6))$, satisfy the identity

$$Jac(RM(2,6), \mathbf{v}_1, X, Z) = Jac(RM(2,6), \mathbf{v}_2, X, Z),$$

and therefore

$$W_{RM(2,6)+\mathbf{v}_1}(X) = W_{RM(2,6)+\mathbf{v}_2}(X).$$

Our strategy to reduce the runtime is to cut off the repetition of the computation with the same outputs. The set of same type vectors is decomposed into the orbits of the action of $\widetilde{K}$. When the weight of the reference vector $\mathbf{v}$ is 8, we look at the types of the distribution of non-zero coordinates of $\mathbf{v}$ into the blocks. The result is

| type | $1^8$ | $2 \cdot 1^6$ | $2^2 \cdot 1^4$ | $2^3 \cdot 1^2$ |
|---|---|---|---|---|
| number of orbits | 4 | 5 | 6 | 3 |
| type | $2^4$ | $3 \cdot 1^5$ | $3 \cdot 2 \cdot 1^3$ | $3 \cdot 2^2 \cdot 1$ |
| number of orbits | 2 | 4 | 4 | 2 |
| type | $3^2 \cdot 1^2$ | $3^2 \cdot 2$ | $4 \cdot 1^4$ | $4 \cdot 2 \cdot 1^2$ |
| number of orbits | 2 | 1 | 3 | 2 |
| type | $4 \cdot 2^2$ | $4 \cdot 3 \cdot 1$ | $4^2$ | |
| number of orbits | 1 | 1 | 1 | |

Here $1^8$ means that non zero coordinates of a $\mathbf{v}$ of weight 8 are distributed into 8 different blocks, and so on. There are four orbits of type $1^8$ under the action of $G$. Each orbit has a certain cardinality. For instance an orbit represented by the block distribution

```
[ 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

has the cardinality 30. This implies that we have only to examine only one case instead of doing 29 other cases with identical outputs. The main body of the program consists of 8 independent cycles. After all we have only to run 41 cases each of which needs small runtime (within 1 minute). In the same way we make programs for other reference vectors of weights of from 9 upto 18.

However when the weights go to more than 12 the runtimes of some cases increase largely. To shorten the runtimes we devise a technical lemma, which will be described in the next subsection.

### 6.2. Group theoretic process II

In many types of block decomposition of the support vector of the reference vector it may contain more than three blocks that contain only one non-zero coordinate. For this case the runtime can be shortened by the following lemma.

**Lemma 1.** *It holds that (i) if the two reference vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same block decomposition with the additional condition that these two contains three (resp. two resp. one) blocks that contain only one non-zero coordinates, then there is an automorphism in $G$ which send $\mathbf{v}_1$ in to a vector whose these three (resp. two resp one) blocks are same one of $\mathbf{v}_2$.*

Proof:
We consider the following permutations for $i \in \{1, 2, \ldots, 16\}$.

$$\begin{aligned} e_1^{(i)} &:= (4(i-1)+1, 4(i-1)+2)(4(i-1)+3, 4(i-1)+4) \\ e_2^{(i)} &:= (4(i-1)+1, 4(i-1)+3)(4(i-1)+2, 4(i-1)+4) \\ e_3^{(i)} &:= (4(i-1)+1, 4(i-1)+4)(4(i-1)+3, 4(i-1)+3) \end{aligned}$$

Let $I$ be a subset of $\{1, 2, \ldots, 16\}$. Using these permutations and $I$, we define the following permutations.

$$f_j^I := \prod_{i \in I} e_j(i)$$

Moreover let $E^I$ denote an elementary abelian permutation group $\{e, f_1^I, f_2^I, f_3^I\}$. The group $E^I$ acts transitively on the block $[4(i-1)+1, 4(i-1)+2, 4(i-1)+3, 4i]$ $(i \in I)$. Let $I_0 := \{1, 2, 3, \ldots, 16\}$, $I_1 := \{1, 2, 3, \ldots, 8\}$, $I_2 := \{1, 2, 3, 4, 9, 10, 11, 12\}$, $I_3 := \{1, 2, 5, 6, 9, 10, 13, 14\}$, $I_4 := \{1, 3, 5, 7, 9, 11, 13, 15\}$. Then it is easy to check that $H := E^{I_0} \times E^{I_1} \times E^{I_2} \times E^{I_3} \times E^{I_4}$ is a subgroup of $G$. In particular $H$ keeps the number of non-zero coordinates in each block.

Let $\{i, j, k\}$ be a subset of $\{1, 2, \ldots, 16\}$ where $(i < j < k)$. Let $v$ be a vectors with length 64 such that the $4(i-1)+1$-th, $4(j-1)+1$-th and $4(k-1)+1$-th constituents are 1 and other constituents are all 0.

For any case of $\{i, j, k\}$, the length of the orbit of v by the action of $H$ is 64. Thus there is an automorphism in $H$ which send $v_1$ in to a vector whose $i$-th, $j$-th and $k$-th block are same one of $v_2$.

By this lemma we can shorten the runtime to 1/64 (resp. 1/16, resp. 1/4) compared to original state. In other word we can cut the number of simple loops by three (resp. two, resp. one) in number. For instance consider the case

```
[ 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0 ]
```

we use 8 simple loops, and by the present lemma we may dispense with 3 simple loops. After these shortenings techniques we realize that the runtimes of finding cosets of weights from 8 upto 18 are largely reduced.

## 7. How to verify the correctness of the computations

We may use the following three features of the distance matrix of the code $RM(2,6)$ to form tests for the correctness of our computations. These features are also useful in detecting unknown cosets of the code. Let $\Psi_{k,j}(X)$ $0 \leq k \leq$

$18, 1 \le j \le u_k$ be the coset weight enumerators of coset weight $k$, and $m_{k,j}$ be the number of the coset of which the coset weight enumerator equals $\Psi_{k,j}(X)$. As to the precise shapes of $\Psi_{k,j}(X)$ the readers may look the tables in the Section 8 and the Appendix.

### 7.1. Combinatorial consideration

This is a simple fact that the totality of vectors of a fixed weight $k$ equals the number $\binom{64}{k}$. For instance the number of the vectors of weight 6 in all the cosets of of weight 6 of $RM(2,6)$ is $55996416 + 1166592 + 17498880 + 312480 = 74974368$ which equals the $\binom{64}{6}$.

If we make a sum :

$$\begin{aligned}
&\Psi_0(X) + 2016\Psi_2(X) + 624960\Psi_{4,1}(X) + 10416\Psi_{4,2}(X) \\
&+55996416\Psi_{6,1}(X) + 1166592\Psi_{6,2}(X) \\
&+17498880\Psi_{6,3}(X) + 312480\Psi_{6,4}(X) \\
&= 1 + 2016X^2 + 635376X^4 + 74974368X^6 \\
&\quad +20852832X^{10} + 550704336X^{12} \\
&\quad +5416215840X^{14} + 25576198956X^{16} \\
&\quad +100780570464X^{18} + 570875930352X^{20} \\
&\quad +2807685692448X^{22} + 9058143866688X^{24} \\
&\quad +20652875068896X^{26} + 37481621234512X^{28} \\
&\quad +55717742982048X^{30} \\
&\quad +64295981751078X^{32} + \cdots
\end{aligned}$$

The coefficients of $X^k$ with $k = 0, 2, 4, 6$ coincide with the binomial coefficients $\binom{64}{k}$. As the upper limit $k$ of the sum grows the polynomial approaches to the binomial expansion of $(1+X)^{64}$

Most general equation is

$$\sum_{k,j} m_{k,j}\Psi_{k,j}(X) = (1+X)^{64}.$$

### 7.2. Delsarte identities

This is the identity (4) in the Section 3.4. Here we give two instances of using it. We note that for the case $\mathbf{C}$ =RM(2,6) the numbers $A_k(\mathbf{C})$, different from 0 in the Section 3.2 are known to be $A_0 = A_{64} = 1, A_{16} = A_{48} = 2604, A_{24} = A_{40} = 291648$

$, A_{28} = A_{36} = 888832, A_{32} = 1828134.$

$$\sum_{i,j}\left(\sum_{\mathbf{e}\in\mathbb{F}_2^n/\mathbf{C}} B_i(\mathbf{e})B_j(\mathbf{e})\right)X^iY^j$$

$$(6) \quad = \sum_{i,j}\left(\sum_k \binom{k}{\frac{(k+i-j)}{2}}\binom{n-k}{\frac{(i+j-k)}{2}}A_k\right)X^iY^j$$

Also by computing (3) we have

$$p_{6,6}^{(0)} = 74974368, p_{6,6}^{(16)} = 0, p_{6,6}^{(24)} = 0, \cdots$$

and

$$p_{6,10}^{(0)} = 0, p_{6,10}^{(16)} = 8008, p_{6,10}^{(24)} = 0, \cdots.$$

When $i = j = 6$, the right hand side of (4) is 74974368 and the lefthand side of (4) is, by a table the Section 8, $55996416 + 1166592 + 17498880 + 312480 = 74974368$.

When $i = 6, j = 10$, the righthand side of (4) is $8008 \cdot 2604 = 20852832$ and the lefthand side of 4 is $1166592 + 17498880 + 7 \cdot 312480 = 20852832$ (See a table in Section 8.)

A more general relation is

$$\begin{aligned}
&55996416\Psi_{6,1}(X)\Psi_{6,1}(Y) + 1166592\Psi_{6,2}(X)\Psi_{6,2}(Y) \\
&+17498880\Psi_{6,3}(X)\Psi_{6,3}(Y) + 312480\Psi_{6,4}(X)\Psi_{6,1}(Y) \\
&= (74974368Y^6 + 20852832Y^{10} + 545965056Y^{12} \\
&\quad +5345907840Y^{14} + 25221719040Y^{16} \\
&\quad +100057012608Y^{18} + 567189697536Y^{20} \\
&\quad +2784073870368Y^{22} + 8976878776320Y^{24} \\
&\quad +20478330740448Y^{26} + 37176664541184Y^{28} \\
&\quad +55248419541504Y^{30} \\
&\quad +63739644401664Y^{32} + \cdots)X^6 + (20852832Y^6 \\
&\quad +33976992Y^{10} + 209986560Y^{12} \\
&\quad +1303666560Y^{14} + 6602910720Y^{16} + 30177985152Y^{18} \\
&\quad +155040076800Y^{20} + 767571309792Y^{22} \\
&\quad +2503357108224Y^{24} + 5737738823712Y^{26} \\
&\quad +10274418395136Y^{28} + 15328932552192Y^{30} \\
&\quad +17852301379584Y^{32} + \cdots)X^{10} + \cdots
\end{aligned}$$

The last terms are a portion of the lefthand side of the equation (5) with $i = 6, j \ge 6$ or $i = 10, j \ge 6$. The portion of the right-hand side of the equation (5) with $i = 6, j \ge 6$ or $i = 10, j \ge 6$ are computed and are verified to be equal to the same last terms.

### 7.3. Partial coincidence

This phenomenon can be well described by the following

**Proposition 3.** *Suppose a binary linear code* $\mathbf{C}$ *is self-orthogonal. If the two reference vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *(possibly of different weight) belong to the same coset* $\mathbf{v}_1 + \mathbf{C} = \mathbf{v}_2 + \mathbf{C}$*, then it holds that* $\mathbf{v}_1 + \mathbf{C}^\perp = \mathbf{v}_2 + \mathbf{C}^\perp$.

The proof is easy, and we leave it for the reader.

A class of codes such as $RM(r,m)$ $r \le \frac{m-1}{2}$ are self-orthogonal, and therefore this proposition can be applied to them.

By this proposition each coset $U$ in $RM(2,6)$ can be naturally regarded as a coset $\varphi(U)$ in $RM(3,6)$. If we can detect which coset of various weights in $RM(2,6)$ corresponds to a coset of certain coset of coset weight $\ell$ in

$RM(3,6)$, the number of the vectors of weight $k$ in these cosets in $RM(2,6)$ should equal the number of the vectors of weight $k$ in the cosets of weight $\ell$.
In $RM(3,6)$ cosets of weight 4 have two different types of coset weight enumerators, and also cosets of weight 6 have two different types of coset weight enumerators. Cosets of other weights in $RM(3,6)$ have unique coset weight enumerator. For $h = 0,1,2,3,5,7,8$ let $I(h,k,j)$ be the set $\{(k,j)\}$ such that $U$ is a coset of weight $k$ and $\varphi(U)$ has the coset weight enumerator $\Phi_h$. For $h = 4,6$ $I(h,1,k,j)$ (resp. $I(h,2,k,j)$) is the set $\{(k,j)\}$ such that $U$ is a coset of weight $k$ and $\varphi(U)$ has the coset weight enumerator $\Phi_{h,1}$ (resp. $\Phi_{h,2}$). We remark that each coefficient of the polynomial

$$\sum_{(k,j)\in I(h,k,j)} m_{k,j}\Psi_{k,j},\ h = 0,1,2,3,5,7,8$$

does not exceed that of the polynomial $n_h\Phi_h(x)$, where $n_h$ is the multiplicity of $\Phi_h(x)$, which is given in the table of Section 5. Similar remark is applied to the case when $h = 4,6$. Using the Table 1 in the Section 8 we observe that the number of all the vectors of weight 8 of the cosets of various weights in $RM(2,6)$ that lead to the coset of weight 2 in $RM(3,6)$ is $2 \cdot 2499840 = 4999680$ while by Table 3 the number of the vectors of weight 8 in the coset of weight 2 in $RM(3,6)$ is $2480 \cdot 2016 = 4999680$. More generally we have

$$\begin{aligned} &2016\Phi_2(X) \\ &\quad = 2016\Psi_2(X) + 312480\Psi_{6,4}(X) + 2499840\Psi_{8,10}(X) \\ &\quad\quad +52496640\Psi_{10,14}(X) + 1749888\Psi_{10,20}(X) \\ &\quad\quad +559964160\Psi_{12,14}(X) + 159989760\Psi_{12,24}(X) \\ &\quad\quad +55996416\Psi_{12,34}(X) + 839946240\Psi_{14,10}(X) \\ &\quad\quad +1749888\Psi_{10,20}(X) + 104993280\Psi_{14,19}(X) \\ &\quad\quad +335978496\Psi_{16,3}(X) \end{aligned}$$

Similar relations hold for $\Phi_0(X), \Phi_1(X)$ etc.

## 8. Explicit Result

We give tables of complete coset weight distributions of RM(2,6). Tables consist of two kind of informations. The Table 3 gives the tabulation of coset weight enumerators of coset weights from 0 upto 18. For each coset weight $i$ the coset weight enumerators $\Psi'$s are indexed as $\Psi_{i,j}$, where $j$ denotes the ordering. When $i = 0,1,2,3,17,18$ the coset weight enumerator is unique and they are simply writtten as $\Psi_i(X)$. When there are many coset weight enumerators for a fixed $i$ we express them as linear combinations of some basis polynomials $\Xi_{i,k}$, to save space.

In the Tables from 4-1 to 4-7 the first column denotes the coset weight enumerator, and the second column implies the coset weight enumerator of $\varphi(U)$ (as to $\varphi$ see Section 7.3), and the third column supplies the multiplicity of the coset with the assigned coset weight enumerator. For instance there are 64 cosets of coset weight 1, and the coset weight distribution of each such coset is desribed by $\Psi_1(X)$.
In some cases there appear cosets $U_1, U_2$ in $RM(2,6)$ of the identical coset weight enumerator $W_{U_1}(X) = W_{U_2}(X)$ but their images $\varphi(U_1), \varphi(U_2)$ in $RM(3,6)$ have different coset weight enumerators $W_{\varphi(U_1)}(X) \neq W_{\varphi(U_2)}(X)$. In the Appendix we give two instances for that phenomenon.

Table 3. Coset Weight Enumerators of $RM(2,6)$ code.

coset weight 0

$$\begin{aligned} \Psi_0(X) &= 1 + 2604X^{16} + 291648X^{24} + 888832X^{28} \\ &\quad +1828134X^{32} + \cdots \end{aligned}$$

coset weight 1

$$\begin{aligned} \Psi_1(X) &= X + 651X^{15} + 1953X^{17} + 109368X^{23} \\ &\quad +182280X^{25} + 388864X^{27} + 499968X^{29} \\ &\quad +914067X^{31} + \cdots \end{aligned}$$

coset weight 2

$$\begin{aligned} \Psi_2(X) &= X^2 + 155X^{14} + 992X^{16} + 1457X^{18} + 39928X^{22} \\ &\quad +138880X^{24} + 279496X^{26} + 444416X^{28} \\ &\quad +727539X^{30} + 928576X^{32} + \cdots \end{aligned}$$

coset weight 3

$$\begin{aligned} \Psi_3(X) &= X^3 + 35X^{13} + 360X^{15} + 1128X^{17} + 1081X^{19} \\ &\quad +14168X^{21} + 77280X^{23} + 200928X^{25} \\ &\quad +359464X^{27} + 593955X^{29} \\ &\quad +848752X^{31} + \cdots \end{aligned}$$

coset weight 4

$$\begin{aligned} \Psi_{4,1}(X) &= X^4 + 7X^{12} + 112X^{14} + 552X^{16} + 1136X^{18} \\ &\quad +5669X^{20} + 37184X^{22} + 127456X^{24} \\ &\quad +273728X^{26} + 478675X^{28} + 736416X^{30} \\ &\quad +872432X^{32} + \cdots \\ \Psi_{4,2}(X) &= X^4 + 35X^{12} + 720X^{16} + 1024X^{18} \\ &\quad +6033X^{20} + 35840X^{22} + 127680X^{24} \\ &\quad +279552X^{26} + 471115X^{28} \\ &\quad +732160X^{30} + 885984X^{32} + \cdots \end{aligned}$$

coset weight 5

$$\begin{aligned}
\Psi_{5,1}(X) &= X^5+7X^{11}+28X^{13}+168X^{15}+808X^{17}\\
&+2724X^{19}+16621X^{21}+71008X^{23}\\
&+192864X^{25}+372939X^{27}\\
&+609152X^{29}+830832X^{31}+\cdots\\
\Psi_{5,2}(X) &= X^5+X^{11}+30X^{13}+220X^{15}+700X^{17}\\
&+2686X^{19}+16863X^{21}+71920X^{23}\\
&+190000X^{25}+372767X^{27}\\
&+616292X^{29}+825672X^{31}+\cdots
\end{aligned}$$

coset weight 6

$$\begin{aligned}
\Psi_{6,1}(X) &= X^6+6X^{12}+75X^{14}+340X^{16}+1311X^{18}\\
&+7566X^{20}+37306X^{22}+119536X^{24}\\
&+272661X^{26}+496748X^{28}\\
&+737222X^{30}+848760X^{32}+\cdots\\
\Psi_{6,2}(X) &= X^6+X^{10}+90X^{14}+320X^{16}+1306X^{18}\\
&+7680X^{20}+37071X^{22}+119552X^{24}\\
&+273231X^{26}+496128X^{28}\\
&+736876X^{30}+849792X^{32}+\cdots\\
\Psi_{6,3}(X) &= X^6+X^{10}+12X^{12}+58X^{14}+328X^{16}\\
&+1402X^{18}+7580X^{20}+36559X^{22}\\
&+120416X^{24}+274511X^{26}+493144X^{28}\\
&+736044X^{30}+854192X^{32}+\cdots\\
\Psi_{6,4}(X) &= X^6+7X^{10}+84X^{14}+224X^{16}+1884X^{18}\\
&+6144X^{20}+38669X^{22}+117376X^{24}\\
&+281323X^{26}+487424X^{28}\\
&+726608X^{30}+874816X^{32}+\cdots
\end{aligned}$$

coset weight 7

$$\begin{aligned}
\Xi_{7,1} &= X^7+3X^{13}+202X^{15}+566X^{17}+3147X^{19}\\
&+18876X^{21}+68726X^{23}+183393X^{25}\\
&+381420X^{27}+627722X^{29}\\
&+813096X^{31}+\cdots\\
\Xi_{7,2} &= X^9-3X^{13}+8X^{15}+28X^{17}-139X^{19}+132X^{21}\\
&-455X^{23}+1358X^{25}+212X^{27}\\
&-4298X^{29}+3156X^{31}+\cdots\\
\Xi_{7,3} &= X^{11}+3X^{13}-16X^{15}+16X^{17}+47X^{19}-187X^{21}\\
&+32X^{23}+608X^{25}-406X^{27}-994X^{29}\\
&+896X^{31}+\cdots\\
\Xi_{7,4} &= 9X^{13}-31X^{15}+9X^{17}+161X^{19}-396X^{21}\\
&+132X^{23}+756X^{25}-764X^{27}-546X^{29}\\
&+670X^{31}+\cdots\\
\Xi_{7,5} &= X^{17}+13X^{19}-121X^{21}+403X^{23}-735X^{25}\\
&+829X^{27}-593X^{29}+203X^{31}+\cdots\\
\Xi_{7,6} &= 35X^{19}-253X^{21}+798X^{23}-1442X^{25}+1645X^{27}\\
&-1203X^{29}+420X^{31}+\cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{7,1}(X) &= \Xi_{7,1}+2\Xi_{7,4}+32\Xi_{7,5}+16\Xi_{7,6}\\
\Psi_{7,2}(X) &= \Xi_{7,1}+\Xi_{7,3}+\Xi_{7,4}\\
\Psi_{7,3}(X) &= \Xi_{7,1}+3\Xi_{7,3}+\Xi_{7,4}\\
\Psi_{7,4}(X) &= \Xi_{7,1}+6\Xi_{7,3}+\Xi_{7,2}\\
\Psi_{7,5}(X) &= \Xi_{7,1}+\Xi_{7,2}+4\Xi_{7,4}\\
\Psi_{7,6}(X) &= \Xi_{7,1}+7\Xi_{7,2}+2\Xi_{7,4}+192\Xi_{7,5}+64\Xi_{7,6}
\end{aligned}$$

coset weight 8

$$\begin{aligned}
\Xi_{8,1} &= X^8-91X^{24}-4256X^{26}+31208X^{28}\\
&-79072X^{30}+107492X^{32}+\cdots\\
\Xi_{8,2} &= X^{10}+26X^{24}+2129X^{26}-1512X^{28}\\
&+686X^{30}+4508X^{32}+\cdots\\
\Xi_{8,3} &= X^{12}-104X^{24}+3232X^{26}-8857X^{28}\\
&+18144X^{30}-19712X^{32}+\cdots\\
\Xi_{8,4} &= X^{14}-14X^{24}+1750X^{26}-5192X^{28}\\
&+11305X^{30}-12628X^{32}+\cdots\\
\Xi_{8,5} &= X^{16}-122X^{24}+1312X^{26}-3912X^{28}\\
&+7776X^{30}-9086X^{32}+\cdots\\
\Xi_{8,6} &= X^{18}-7X^{24}+266X^{26}-724X^{28}\\
&+1589X^{30}-1738X^{32}+\cdots\\
\Xi_{8,7} &= X^{20}-46X^{24}+256X^{26}-721X^{28}\\
&+1280X^{30}-1540X^{32}+\cdots\\
\Xi_{8,8} &= X^{22}-161X^{24}+749X^{26}-2252X^{28}\\
&+3794X^{30}-4774X^{32}+\cdots\\
\Xi_{8,9} &= 455X^{24}-2128X^{26}+6452X^{28}-10864X^{30}\\
&+13706X^{32}+\cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{8,1}(X) &= \Xi_{8,1} + 56\Xi_{8,4} + 266\Xi_{8,5} + 1400\Xi_{8,6} \\
&\quad +8512\Xi_{8,7} + 37216\Xi_{8,8} + 14378\Xi_{8,9} \\
\Psi_{8,2}(X) &= \Xi_{8,1} + \Xi_{8,3} + 50\Xi_{8,4} + 276\Xi_{8,5} \\
&\quad +1410\Xi_{8,6} + 8461\Xi_{8,7} + 37256\Xi_{8,8} \\
&\quad +14390\Xi_{8,9} \\
\Psi_{8,3}(X) &= \Xi_{8,1} + 6\Xi_{8,3} + 44\Xi_{8,4} + 254\Xi_{8,5} \\
&\quad +1484\Xi_{8,6} + 8398\Xi_{8,7} + 36976\Xi_{8,8} \\
&\quad +14282\Xi_{8,9} \\
\Psi_{8,4}(X) &= \Xi_{8,1} + 8\Xi_{8,3} + 32\Xi_{8,4} + 282\Xi_{8,5} \\
&\quad +1440\Xi_{8,6} + 8488\Xi_{8,7} + 36864\Xi_{8,8} \\
&\quad +14258\Xi_{8,9} \\
\Psi_{8,5}(X) &= \Xi_{8,1} + 12\Xi_{8,3} + 32\Xi_{8,4} + 242\Xi_{8,5} \\
&\quad +1568\Xi_{8,6} + 8284\Xi_{8,7} + 36736\Xi_{8,8} \\
&\quad +14186\Xi_{8,9} \\
\Psi_{8,6}(X) &= \Xi_{8,1} + \Xi_{8,2} + 6\Xi_{8,3} + 47\Xi_{8,4} \\
&\quad +222\Xi_{8,5} + 1563\Xi_{8,6} + 8398\Xi_{8,7} \\
&\quad +36501\Xi_{8,8} + 14108\Xi_{8,9} \\
\Psi_{8,7}(X) &= \Xi_{8,1} + 2\Xi_{8,2} + 8\Xi_{8,3} + 38\Xi_{8,4} \\
&\quad +266\Xi_{8,5} + 1470\Xi_{8,6} + 8360\Xi_{8,7} \\
&\quad +36554\Xi_{8,8} + 14134\Xi_{8,9} \\
\Psi_{8,8}(X) &= 2\Xi_{8,1} + 12\Xi_{8,3} + 48\Xi_{8,4} + 220\Xi_{8,5} \\
&\quad +1584\Xi_{8,6} + 8220\Xi_{8,7} + 35904\Xi_{8,8} \\
&\quad +13884\Xi_{8,9} \\
\Psi_{8,9}(X) &= 2\Xi_{8,1} + 24\Xi_{8,3} + 420\Xi_{8,5} \\
&\quad +1024\Xi_{8,6} + 8888\Xi_{8,7} + 35840\Xi_{8,8} \\
&\quad +13972\Xi_{8,9} \\
\Psi_{8,10}(X) &= 2\Xi_{8,1} + 6\Xi_{8,2} + 42\Xi_{8,4} + 308\Xi_{8,5} \\
&\quad +1778\Xi_{8,6} + 6272\Xi_{8,7} + 39934\Xi_{8,8} \\
&\quad +15128\Xi_{8,9} \\
\Psi_{8,11}(X) &= 8\Xi_{8,1} + 784\Xi_{8,5} + 14336\Xi_{8,7} + 2192\Xi_{8,9}
\end{aligned}$$

coset weight 9

$$\begin{aligned}
\Xi_{9,1} &= X^9 - 451X^{21} - 520X^{23} + 161X^{25} - 3919X^{27} \\
&\quad -8030X^{29} - 3626X^{31} + \cdots \\
\Xi_{9,2} &= X^{11} - 246059X^{21} + 171080X^{23} - 1018872X^{25} \\
&\quad -2164078X^{27} - 1143144X^{29} - 3987536X^{31} + \cdots \\
\Xi_{9,3} &= X^{13} - 1067X^{21} + 988X^{23} - 4172X^{25} - 8913X^{27} \\
&\quad -3589X^{29} - 16016X^{31} + \cdots \\
\Xi_{9,4} &= X^{15} + X^{19} + 418X^{21} - 142X^{23} + 1848X^{25} \\
&\quad +3964X^{27} + 2833X^{29} + 7461X^{31} + \cdots \\
\Xi_{9,5} &= X^{17} - 22X^{21} + 65X^{23} - 49X^{25} - 86X^{27} \\
&\quad +204X^{29} - 113X^{31} + \cdots \\
\Xi_{9,6} &= 3X^{19} - 61501X^{21} + 42750X^{23} - 254786X^{25} \\
&\quad -540947X^{27} - 285651X^{29} - 997020X^{31} + \cdots \\
\Xi_{9,7} &= 1441X^{21} - 1001X^{23} + 5971X^{25} + 12677X^{27} \\
&\quad +6698X^{29} + 23366X^{31} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{9,1}(X) &= \Xi_{9,1} + 3\Xi_{9,3} + 126\Xi_{9,4} + 594\Xi_{9,5} \\
&\quad +1167\Xi_{9,6} + 49795\Xi_{9,7} \\
\Psi_{9,2}(X) &= \Xi_{9,1} + 11\Xi_{9,3} + 106\Xi_{9,4} + 590\Xi_{9,5} \\
&\quad +1203\Xi_{9,6} + 51343\Xi_{9,7} \\
\Psi_{9,3}(X) &= \Xi_{9,1} + 12\Xi_{9,3} + 99\Xi_{9,4} + 607\Xi_{9,5} \\
&\quad +1203\Xi_{9,6} + 51346\Xi_{9,7} \\
\Psi_{9,4}(X) &= \Xi_{9,1} + 20\Xi_{9,3} + 75\Xi_{9,4} + 631\Xi_{9,5} \\
&\quad +1219\Xi_{9,6} + 52042\Xi_{9,7} \\
\Psi_{9,5}(X) &= \Xi_{9,1} + \Xi_{9,2} + 14\Xi_{9,3} + 90\Xi_{9,4} \\
&\quad +606\Xi_{9,5} + 1224\Xi_{9,6} + 52417\Xi_{9,7} \\
\Psi_{9,6}(X) &= \Xi_{9,1} + 3\Xi_{9,2} + 13\Xi_{9,3} + 79\Xi_{9,4} \\
&\quad +603\Xi_{9,5} + 1270\Xi_{9,6} + 54724\Xi_{9,7} \\
\Psi_{9,7}(X) &= \Xi_{9,1} + 3\Xi_{9,2} + 13\Xi_{9,3} + 95\Xi_{9,4} \\
&\quad +603\Xi_{9,5} + 1206\Xi_{9,6} + 51988\Xi_{9,7} \\
\Psi_{9,8}(X) &= \Xi_{9,1} + 3\Xi_{9,2} + 17\Xi_{9,3} + 67\Xi_{9,4} \\
&\quad +687\Xi_{9,5} + 1158\Xi_{9,6} + 49952\Xi_{9,7} \\
\Psi_{9,9}(X) &= \Xi_{9,1} + 5\Xi_{9,2} + 11\Xi_{9,3} + 87\Xi_{9,4} \\
&\quad +611\Xi_{9,5} + 1232\Xi_{9,6} + 53440\Xi_{9,7} \\
\Psi_{9,10}(X) &= 2 * \Xi_{9,1} + 2\Xi_{9,2} + 18\Xi_{9,3} \\
&\quad +90\Xi_{9,4} + 594\Xi_{9,5} + 1232\Xi_{9,6} + 52932\Xi_{9,7} \\
\Psi_{9,11}(X) &= 2 * \Xi_{9,1} + 4\Xi_{9,2} + 20\Xi_{9,3} \\
&\quad +102\Xi_{9,4} + 574\Xi_{9,5} + 1162\Xi_{9,6} + 50284\Xi_{9,7} \\
\Psi_{9,12}(X) &= 3 * \Xi_{9,1} + 5\Xi_{9,2} + 7\Xi_{9,3} \\
&\quad +105\Xi_{9,4} + 749\Xi_{9,5} + 1022\Xi_{9,6} + 44472\Xi_{9,7}
\end{aligned}$$

coset weight 10

$$\begin{aligned}
\Xi_{10,1} &= X^{10} + 37X^{22} - 16X^{24} + 2178X^{26} - 960X^{28} \\ &\quad -168X^{30} + 6048X^{32} + \cdots \\
\Xi_{10,2} &= X^{12} + 73X^{22} + 428X^{24} + 453X^{26} + 951X^{28} \\ &\quad +1778X^{30} + 1848X^{32} + \cdots \\
\Xi_{10,3} &= X^{14} - 183X^{22} + 784X^{24} - 1253X^{26} \\ &\quad +448X^{28} + 1435X^{30} - 2464X^{32} + \cdots \\
\Xi_{10,4} &= X^{16} + 84X^{22} + 4X^{24} + 388X^{26} + 480X^{28} \\ &\quad +552X^{30} + 1078X^{32} + \cdots \\
\Xi_{10,5} &= X^{18} - 29X^{22} + 112X^{24} - 175X^{26} + 64X^{28} \\ &\quad +203X^{30} - 352X^{32} + \cdots \\
\Xi_{10,6} &= X^{20} + 996X^{22} - 2062X^{24} + 5716X^{26} \\ &\quad +1583X^{28} - 568X^{30} + 13244X^{32} + \cdots \\
\Xi_{10,7} &= 209X^{22} - 434X^{24} + 1197X^{26} + 328X^{28} \\ &\quad -126X^{30} + 2772X^{32} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{10,1}(X) &= \Xi_{10,1} + 27\Xi_{10,3} + 264\Xi_{10,4} \\ &\quad +1511\Xi_{10,5} + 8768\Xi_{10,6} - 41480\Xi_{10,7} \\
\Psi_{10,2}(X) &= \Xi_{10,1} + 39\Xi_{10,3} + 208\Xi_{10,4} \\ &\quad +1619\Xi_{10,5} + 8704\Xi_{10,6} - 41128\Xi_{10,7} \\
\Psi_{10,3}(X) &= \Xi_{10,1} + \Xi_{10,2} + 31\Xi_{10,3} \\ &\quad +238\Xi_{10,4} + 1547\Xi_{10,5} + 8813\Xi_{10,6} \\ &\quad -41677\Xi_{10,7} \\
\Psi_{10,4}(X) &= \Xi_{10,1} + 2\Xi_{10,2} + 33\Xi_{10,3} \\ &\quad +224\Xi_{10,4} + 1565\Xi_{10,5} + 8826\Xi_{10,6} \\ &\quad -41730\Xi_{10,7} \\
\Psi_{10,5}(X) &= \Xi_{10,1} + 3\Xi_{10,2} + 27\Xi_{10,3} \\ &\quad +234\Xi_{10,4} + 1575\Xi_{10,5} + 8775\Xi_{10,6} \\ &\quad -41495\Xi_{10,7} \\
\Psi_{10,6}(X) &= \Xi_{10,1} + 3\Xi_{10,2} + 29\Xi_{10,3} \\ &\quad +222\Xi_{10,4} + 1593\Xi_{10,5} + 8807\Xi_{10,6} \\ &\quad -41639\Xi_{10,7} \\
\Psi_{10,7}(X) &= \Xi_{10,1} + 4\Xi_{10,2} + 31\Xi_{10,3} \\ &\quad +232\Xi_{10,4} + 1547\Xi_{10,5} + 8756\Xi_{10,6} \\ &\quad -41404\Xi_{10,7} \\
\Psi_{10,8}(X) &= \Xi_{10,1} + 4\Xi_{10,2} + 39\Xi_{10,3} \\ &\quad +200\Xi_{10,4} + 1619\Xi_{10,5} + 8628\Xi_{10,6} \\ &\quad -40764\Xi_{10,7} \\
\Psi_{10,9}(X) &= \Xi_{10,1} + 6\Xi_{10,2} + 27\Xi_{10,3} \\ &\quad +228\Xi_{10,4} + 1575\Xi_{10,5} + 8718\Xi_{10,6} \\ &\quad -41222\Xi_{10,7} \\
\Psi_{10,10}(X) &= \Xi_{10,1} + 10\Xi_{10,2} + 15\Xi_{10,3} \\ &\quad +172\Xi_{10,4} + 1787\Xi_{10,5} + 8770\Xi_{10,6} \\ &\quad -41434\Xi_{10,7}
\end{aligned}$$

$$\begin{aligned}
\Psi_{10,11}(X) &= 2\Xi_{10,1} + 4\Xi_{10,2} + 34\Xi_{10,3} \\ &\quad +224\Xi_{10,4} + 1562\Xi_{10,5} + 8692\Xi_{10,6} \\ &\quad -41092\Xi_{10,7} \\
\Psi_{10,12}(X) &= 2\Xi_{10,1} + 8\Xi_{10,2} + 22\Xi_{10,3} \\ &\quad +224\Xi_{10,4} + 1582\Xi_{10,5} + 8808\Xi_{10,6} \\ &\quad -41656\Xi_{10,7} \\
\Psi_{10,13}(X) &= 2\Xi_{10,1} + 8\Xi_{10,2} + 38\Xi_{10,3} \\ &\quad +224\Xi_{10,4} + 1470\Xi_{10,5} + 8808\Xi_{10,6} \\ &\quad -41656\Xi_{10,7} \\
\Psi_{10,14}(X) &= 2\Xi_{10,1} + 8\Xi_{10,2} + 46\Xi_{10,3} \\ &\quad +144\Xi_{10,4} + 1862\Xi_{10,5} + 7528\Xi_{10,6} \\ &\quad -35448\Xi_{10,7} \\
\Psi_{10,15}(X) &= 3\Xi_{10,1} + 6\Xi_{10,2} + 29\Xi_{10,3} \\ &\quad +212\Xi_{10,4} + 1633\Xi_{10,5} + 8590\Xi_{10,6} \\ &\quad -40598\Xi_{10,7} \\
\Psi_{10,16}(X) &= 4\Xi_{10,1} + 44\Xi_{10,3} + 192\Xi_{10,4} \\ &\quad +1628\Xi_{10,5} + 8704\Xi_{10,6} - 41120\Xi_{10,7} \\
\Psi_{10,17}(X) &= 4\Xi_{10,1} + 4\Xi_{10,2} + 28\Xi_{10,3} \\ &\quad +280\Xi_{10,4} + 1484\Xi_{10,5} + 8372\Xi_{10,6} \\ &\quad -39604\Xi_{10,7} \\
\Psi_{10,18}(X) &= 4\Xi_{10,1} + 4\Xi_{10,2} + 44\Xi_{10,3} \\ &\quad +216\Xi_{10,4} + 1500\Xi_{10,5} + 8756\Xi_{10,6} \\ &\quad -41396\Xi_{10,7} \\
\Psi_{10,19}(X) &= 4\Xi_{10,1} + 8\Xi_{10,2} + 12\Xi_{10,3} \\ &\quad +304\Xi_{10,4} + 1468\Xi_{10,5} + 8424\Xi_{10,6} \\ &\quad -39880\Xi_{10,7} \\
\Psi_{10,20}(X) &= 6\Xi_{10,1} + 106\Xi_{10,3} + 32\Xi_{10,4} \\ &\quad +1842\Xi_{10,5} + 7680\Xi_{10,6} - 36080\Xi_{10,7}
\end{aligned}$$

coset weight 11

$$\begin{aligned}
\Xi_{11,1} &= X^{11}+3X^{19}+61336X^{21}-42426X^{23} \\
&\quad +254918X^{25}+540287X^{27}+285893X^{29} \\
&\quad +997140X^{31}+\cdots \\
\Xi_{11,2} &= X^{13}-3X^{19}-88X^{21}+280X^{23}-168X^{25} \\
&\quad -400X^{27}+746X^{29}-368X^{31}+\cdots \\
\Xi_{11,3} &= X^{15}-47X^{19}+231X^{21}-459X^{23} \\
&\quad +231X^{25}+725X^{27}-1485X^{29}+803X^{31}+\cdots \\
\Xi_{11,4} &= X^{17}-9X^{19}+33X^{21}-57X^{23}+21X^{25} \\
&\quad +99X^{27}-187X^{29}+99X^{31}+\cdots \\
\Xi_{11,5} &= X^{19}-297X^{21}+9700X^{23}-33460X^{25} \\
&\quad +84093X^{27}-82645X^{29}+55376X^{31}+\cdots \\
\Xi_{11,6} &= 11X^{21}+3549X^{23}-12047X^{25}+32599X^{27} \\
&\quad -30450X^{29}+22722X^{31}+\cdots \\
\Xi_{11,7} &= 455X^{23}-1547X^{25}+4158X^{27}-3902X^{29} \\
&\quad +2884X^{31}+\cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{11,1}(X) &= \Xi_{11,1}+3*\Xi_{11,2}+84*\Xi_{11,3}+660*\Xi_{11,4} \\
&\quad +13653\Xi_{11,5}+361046\Xi_{11,6}-3106816\Xi_{11,7} \\
\Psi_{11,2}(X) &= \Xi_{11,1}+4\Xi_{11,2}+85\Xi_{11,3}+621\Xi_{11,4} \\
&\quad +13473\Xi_{11,5}+356286\Xi_{11,6}-3065856\Xi_{11,7} \\
\Psi_{11,3}(X) &= \Xi_{11,1}+5\Xi_{11,2}+82\Xi_{11,3}+610\Xi_{11,4} \\
&\quad +13293\Xi_{11,5}+351526\Xi_{11,6}-3024896\Xi_{11,7} \\
\Psi_{11,4}(X) &= \Xi_{11,1}+6\Xi_{11,2}+75\Xi_{11,3}+627\Xi_{11,4} \\
&\quad +13113\Xi_{11,5}+346766\Xi_{11,6}-2983936\Xi_{11,7} \\
\Psi_{11,5}(X) &= \Xi_{11,1}+7\Xi_{11,2}+88\Xi_{11,3}+616\Xi_{11,4} \\
&\quad +13509\Xi_{11,5}+357238\Xi_{11,6}-3074048\Xi_{11,7} \\
\Psi_{11,6}(X) &= \Xi_{11,1}+7\Xi_{11,2}+96\Xi_{11,3}+544\Xi_{11,4} \\
&\quad +13509\Xi_{11,5}+357238\Xi_{11,6}-3074048\Xi_{11,7} \\
\Psi_{11,7}(X) &= \Xi_{11,1}+9\Xi_{11,2}+78\Xi_{11,3}+622\Xi_{11,4} \\
&\quad +13149\Xi_{11,5}+347718\Xi_{11,6}-2992128\Xi_{11,7} \\
\Psi_{11,8}(X) &= \Xi_{11,1}+10\Xi_{11,2}+55\Xi_{11,3}+639\Xi_{11,4} \\
&\quad +12393\Xi_{11,5}+327726\Xi_{11,6}-2820096\Xi_{11,7} \\
\Psi_{11,9}(X) &= \Xi_{11,1}+10\Xi_{11,2}+75\Xi_{11,3}+611\Xi_{11,4} \\
&\quad +12969\Xi_{11,5}+342958\Xi_{11,6}-2951168\Xi_{11,7} \\
\Psi_{11,10}(X) &= \Xi_{11,1}+11\Xi_{11,2}+68\Xi_{11,3}+628\Xi_{11,4} \\
&\quad +12789\Xi_{11,5}+338198\Xi_{11,6}-2910208\Xi_{11,7} \\
\Psi_{11,11}(X) &= \Xi_{11,1}+15\Xi_{11,2}+64\Xi_{11,3}+640\Xi_{11,4} \\
&\quad +12645\Xi_{11,5}+334390\Xi_{11,6}-2877440\Xi_{11,7} \\
\Psi_{11,12}(X) &= 2\Xi_{11,1}+8\Xi_{11,2}+86\Xi_{11,3}+598\Xi_{11,4} \\
&\quad +13314\Xi_{11,5}+346492\Xi_{11,6}-2985984\Xi_{11,7} \\
\Psi_{11,13}(X) &= 2\Xi_{11,1}+9\Xi_{11,2}+79\Xi_{11,3}+615\Xi_{11,4} \\
&\quad +13134\Xi_{11,5}+341732\Xi_{11,6}-2945024\Xi_{11,7} \\
\Psi_{11,14}(X) &= 2\Xi_{11,1}+10\Xi_{11,2}+76\Xi_{11,3}+604\Xi_{11,4} \\
&\quad +12954\Xi_{11,5}+336972\Xi_{11,6}-2904064\Xi_{11,7} \\
\Psi_{11,15}(X) &= 2\Xi_{11,1}+18\Xi_{11,2}+68\Xi_{11,3}+644\Xi_{11,4} \\
&\quad +12666\Xi_{11,5}+329356\Xi_{11,6}+365998080\Xi_{11,7} \\
\Psi_{11,16}(X) &= 2\Xi_{11,1}+18\Xi_{11,2}+76\Xi_{11,3}+572\Xi_{11,4} \\
&\quad +12666\Xi_{11,5}+329356\Xi_{11,6}-2838528\Xi_{11,7} \\
\Psi_{11,17}(X) &= 3\Xi_{11,1}+10\Xi_{11,2}+73\Xi_{11,3}+625\Xi_{11,4} \\
&\quad +12939\Xi_{11,5}+330986\Xi_{11,6}-2856960\Xi_{11,7} \\
\Psi_{11,18}(X) &= 4\Xi_{11,1}+4\Xi_{11,2}+88\Xi_{11,3}+600\Xi_{11,4} \\
&\quad +13428\Xi_{11,5}+338328\Xi_{11,6}-2924544\Xi_{11,7} \\
\Psi_{11,19}(X) &= 4\Xi_{11,1}+12\Xi_{11,2}+88\Xi_{11,3}+568\Xi_{11,4} \\
&\quad +13140\Xi_{11,5}+330712\Xi_{11,6}-2859008\Xi_{11,7} \\
\Psi_{11,20}(X) &= 4\Xi_{11,1}+16\Xi_{11,2}+28\Xi_{11,3}+636\Xi_{11,4} \\
&\quad +11268\Xi_{11,5}+281208\Xi_{11,6}-2433024\Xi_{11,7} \\
\Psi_{11,21}(X) &= 5\Xi_{11,1}+3\Xi_{11,2}+112\Xi_{11,3}+560\Xi_{11,4} \\
&\quad +14169\Xi_{11,5}+352334\Xi_{11,6}-3049472\Xi_{11,7} \\
\Psi_{11,22}(X) &= 5\Xi_{11,1}+7\Xi_{11,2}+84\Xi_{11,3}+644\Xi_{11,4} \\
&\quad +13449\Xi_{11,5}+333294\Xi_{11,6}-2885632\Xi_{11,7} \\
\Psi_{11,23}(X) &= 6\Xi_{11,1}+10\Xi_{11,2}+56\Xi_{11,3}+632\Xi_{11,4} \\
&\quad +12318\Xi_{11,5}+297796\Xi_{11,6}-2584576\Xi_{11,7} \\
\Psi_{11,24}(X) &= 6\Xi_{11,1}+10\Xi_{11,2}+80\Xi_{11,3}+592\Xi_{11,4} \\
&\quad +12894\Xi_{11,5}+313028\Xi_{11,6}-2715648\Xi_{11,7} \\
\Psi_{11,25}(X) &= 6\Xi_{11,1}+26\Xi_{11,2}+80\Xi_{11,3}+528\Xi_{11,4} \\
&\quad +12318\Xi_{11,5}+297796\Xi_{11,6}-2584576\Xi_{11,7} \\
\Psi_{11,26}(X) &= 6\Xi_{11,1}+26\Xi_{11,2}+96\Xi_{11,3}+288\Xi_{11,4} \\
&\quad +12574\Xi_{11,5}+304452\Xi_{11,6}-2641920\Xi_{11,7}
\end{aligned}$$

coset weight 12

$$\begin{aligned}
\Xi_{12,1} &= X^{12} - 1924X^{24} + 11744X^{26} - 34665X^{28} \\ &\quad +61600X^{30} - 74536X^{32} + \cdots \\
\Xi_{12,2} &= X^{14} - 924X^{24} + 6006X^{26} - 18096X^{28} \\ &\quad +33033X^{30} - 40040X^{32} + \cdots \\
\Xi_{12,3} &= X^{16} - 1032X^{24} + 5568X^{26} - 16816X^{28} \\ &\quad +29504X^{30} - 36498X^{32} + \cdots \\
\Xi_{12,4} &= X^{18} + X^{22} - 168X^{24} + 1015X^{26} - 2976X^{28} \\ &\quad +5383X^{30} - 6512X^{32} + \cdots \\
\Xi_{12,5} &= X^{20} - 46X^{24} + 256X^{26} - 721X^{28} + 1280X^{30} \\ &\quad -1540X^{32} + \cdots \\
\Xi_{12,6} &= 3X^{20} - 28X^{24} + 119X^{26} - 304X^{28} + 518X^{30} \\ &\quad -616X^{32} + \cdots \\
\Xi_{12,7} &= 455X^{24} - 2128X^{26} + 6452X^{28} - 10864X^{30} \\ &\quad +13706X^{32} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Psi_{12,1}(X) &= \Xi_{12,1} + 12\Xi_{12,2} + 254\Xi_{12,3} + 1580\Xi_{12,4} \\ &\quad +9069\Xi_{12,5} + 36464\Xi_{12,6} - 20186\Xi_{12,7} \\
\Psi_{12,2}(X) &= \Xi_{12,1} + 16\Xi_{12,2} + 222\Xi_{12,3} + 1680\Xi_{12,4} \\ &\quad +8941\Xi_{12,5} + 36416\Xi_{12,6} - 20130\Xi_{12,7} \\
\Psi_{12,3}(X) &= \Xi_{12,1} + 18\Xi_{12,2} + 234\Xi_{12,3} + 1634\Xi_{12,4} \\ &\quad +8909\Xi_{12,5} + 36616\Xi_{12,6} - 20278\Xi_{12,7} \\
\Psi_{12,4}(X) &= \Xi_{12,1} + 20\Xi_{12,2} + 222\Xi_{12,3} + 1652\Xi_{12,4} \\ &\quad +8941\Xi_{12,5} + 36496\Xi_{12,6} - 20202\Xi_{12,7} \\
\Psi_{12,5}(X) &= \Xi_{12,1} + 22\Xi_{12,2} + 210\Xi_{12,3} + 1670\Xi_{12,4} \\ &\quad +8973\Xi_{12,5} + 36376\Xi_{12,6} - 20126\Xi_{12,7} \\
\Psi_{12,6}(X) &= \Xi_{12,1} + 24\Xi_{12,2} + 222\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8941\Xi_{12,5} + 36576\Xi_{12,6} - 20274\Xi_{12,7} \\
\Psi_{12,7}(X) &= 2\Xi_{12,1} + 20\Xi_{12,2} + 244\Xi_{12,3} + 1588\Xi_{12,4} \\ &\quad +8858\Xi_{12,5} + 36816\Xi_{12,6} - 20428\Xi_{12,7} \\
\Psi_{12,8}(X) &= 2\Xi_{12,1} + 24\Xi_{12,2} + 220\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8922\Xi_{12,5} + 36576\Xi_{12,6} - 20276\Xi_{12,7} \\
\Psi_{12,9}(X) &= 2\Xi_{12,1} + 24\Xi_{12,2} + 236\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8666\Xi_{12,5} + 37088\Xi_{12,6} - 20596\Xi_{12,7} \\
\Psi_{12,10}(X) &= 2\Xi_{12,1} + 24\Xi_{12,2} + 268\Xi_{12,3} + 1304\Xi_{12,4} \\ &\quad +10074\Xi_{12,5} + 33568\Xi_{12,6} - 18420\Xi_{12,7} \\
\Psi_{12,11}(X) &= 2\Xi_{12,1} + 28\Xi_{12,2} + 212\Xi_{12,3} + 1660\Xi_{12,4} \\ &\quad +8730\Xi_{12,5} + 36848\Xi_{12,6} - 20444\Xi_{12,7} \\
\Psi_{12,12}(X) &= 2\Xi_{12,1} + 28\Xi_{12,2} + 220\Xi_{12,3} + 1596\Xi_{12,4} \\ &\quad +8922\Xi_{12,5} + 36656\Xi_{12,6} - 20348\Xi_{12,7} \\
\Psi_{12,13}(X) &= 2\Xi_{12,1} + 32\Xi_{12,2} + 188\Xi_{12,3} + 1696\Xi_{12,4} \\ &\quad +8794\Xi_{12,5} + 36608\Xi_{12,6} - 20292\Xi_{12,7} \\
\Psi_{12,14}(X) &= 3\Xi_{12,1} + 24\Xi_{12,2} + 218\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8903\Xi_{12,5} + 36576\Xi_{12,6} - 20278\Xi_{12,7} \\
\Psi_{12,15}(X) &= 4\Xi_{12,1} + 312\Xi_{12,3} + 1536\Xi_{12,4} \\ &\quad +8628\Xi_{12,5} + 37376\Xi_{12,6} - 20744\Xi_{12,7} \\
\Psi_{12,16}(X) &= 4\Xi_{12,1} + 16\Xi_{12,2} + 248\Xi_{12,3} + 1552\Xi_{12,4} \\ &\quad +9012\Xi_{12,5} + 36544\Xi_{12,6} - 20264\Xi_{12,7} \\
\Psi_{12,17}(X) &= 4\Xi_{12,1} + 20\Xi_{12,2} + 192\Xi_{12,3} + 1716\Xi_{12,4} \\ &\quad +8948\Xi_{12,5} + 36176\Xi_{12,6} - 19984\Xi_{12,7} \\
\Psi_{12,18}(X) &= 4\Xi_{12,1} + 22\Xi_{12,2} + 228\Xi_{12,3} + 1606\Xi_{12,4} \\ &\quad +8852\Xi_{12,5} + 36696\Xi_{12,6} - 20356\Xi_{12,7} \\
\Psi_{12,19}(X) &= 4\Xi_{12,1} + 24\Xi_{12,2} + 216\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8884\Xi_{12,5} + 36576\Xi_{12,6} - 20280\Xi_{12,7} \\
\Psi_{12,20}(X) &= 4\Xi_{12,1} + 32\Xi_{12,2} + 216\Xi_{12,3} + 1568\Xi_{12,4} \\ &\quad +8884\Xi_{12,5} + 36736\Xi_{12,6} - 20424\Xi_{12,7} \\
\Psi_{12,21}(X) &= 4\Xi_{12,1} + 32\Xi_{12,2} + 248\Xi_{12,3} + 1440\Xi_{12,4} \\ &\quad +9012\Xi_{12,5} + 36864\Xi_{12,6} - 20552\Xi_{12,7} \\
\Psi_{12,22}(X) &= 4\Xi_{12,1} + 36\Xi_{12,2} + 192\Xi_{12,3} + 1604\Xi_{12,4} \\ &\quad +8948\Xi_{12,5} + 36496\Xi_{12,6} - 20272\Xi_{12,7} \\
\Psi_{12,23}(X) &= 5\Xi_{12,1} + 18\Xi_{12,2} + 226\Xi_{12,3} + 1634\Xi_{12,4} \\ &\quad +8833\Xi_{12,5} + 36616\Xi_{12,6} - 20286\Xi_{12,7} \\
\Psi_{12,24}(X) &= 6\Xi_{12,1} + 16\Xi_{12,2} + 308\Xi_{12,3} + 1232\Xi_{12,4} \\ &\quad +9870\Xi_{12,5} + 34048\Xi_{12,6} - 18732\Xi_{12,7} \\
\Psi_{12,25}(X) &= 6\Xi_{12,1} + 20\Xi_{12,2} + 212\Xi_{12,3} + 1652\Xi_{12,4} \\ &\quad +8846\Xi_{12,5} + 36496\Xi_{12,6} - 20212\Xi_{12,7} \\
\Psi_{12,26}(X) &= 6\Xi_{12,1} + 20\Xi_{12,2} + 236\Xi_{12,3} + 1588\Xi_{12,4} \\ &\quad +8782\Xi_{12,5} + 36816\Xi_{12,6} - 20436\Xi_{12,7} \\
\Psi_{12,27}(X) &= 6\Xi_{12,1} + 24\Xi_{12,2} + 228\Xi_{12,3} + 1624\Xi_{12,4} \\ &\quad +8590\Xi_{12,5} + 37088\Xi_{12,6} - 20604\Xi_{12,7} \\
\Psi_{12,28}(X) &= 6\Xi_{12,1} + 24\Xi_{12,2} + 244\Xi_{12,3} + 1496\Xi_{12,4} \\ &\quad +8974\Xi_{12,5} + 36704\Xi_{12,6} - 20412\Xi_{12,7} \\
\Psi_{12,29}(X) &= 6\Xi_{12,1} + 28\Xi_{12,2} + 212\Xi_{12,3} + 1596\Xi_{12,4} \\ &\quad +8846\Xi_{12,5} + 36656\Xi_{12,6} - 20356\Xi_{12,7} \\
\Psi_{12,30}(X) &= 6\Xi_{12,1} + 28\Xi_{12,2} + 220\Xi_{12,3} + 1532\Xi_{12,4} \\ &\quad +9038\Xi_{12,5} + 36464\Xi_{12,6} - 20260\Xi_{12,7} \\
\Psi_{12,31}(X) &= 8\Xi_{12,1} + 16\Xi_{12,2} + 240\Xi_{12,3} + 1552\Xi_{12,4} \\ &\quad +8936\Xi_{12,5} + 36544\Xi_{12,6} - 20272\Xi_{12,7} \\
\Psi_{12,32}(X) &= 8\Xi_{12,1} + 32\Xi_{12,2} + 208\Xi_{12,3} + 1568\Xi_{12,4} \\ &\quad +8808\Xi_{12,5} + 36736\Xi_{12,6} - 20432\Xi_{12,7} \\
\Psi_{12,33}(X) &= 12\Xi_{12,1} + 16\Xi_{12,2} + 136\Xi_{12,3} + 1808\Xi_{12,4} \\ &\quad +9116\Xi_{12,5} + 35264\Xi_{12,6} - 19384\Xi_{12,7} \\
\Psi_{12,34}(X) &= 12\Xi_{12,1} + 40\Xi_{12,2} + 152\Xi_{12,3} + 1832\Xi_{12,4} \\ &\quad +7580\Xi_{12,5} + 39904\Xi_{12,6} - 22440\Xi_{12,7} \\
\Psi_{12,35}(X) &= 12\Xi_{12,1} + 40\Xi_{12,2} + 184\Xi_{12,3} + 1512\Xi_{12,4} \\ &\quad +8988\Xi_{12,5} + 36384\Xi_{12,6} - 20264\Xi_{12,7} \\
\Psi_{12,36}(X) &= 16\Xi_{12,1} + 192\Xi_{12,3} + 1792\Xi_{12,4} \\ &\quad +8656\Xi_{12,5} + 36096\Xi_{12,6} - 19872\Xi_{12,7} \\
\Psi_{12,37}(X) &= 16\Xi_{12,1} + 352\Xi_{12,3} + 1024\Xi_{12,4} \\ &\quad +9936\Xi_{12,5} + 35840\Xi_{12,6} - 20000\Xi_{12,7} \\
\Psi_{12,38}(X) &= 32\Xi_{12,1} + 320\Xi_{12,3} + 1024\Xi_{12,4} \\ &\quad +9632\Xi_{12,5} + 35840\Xi_{12,6} - 20032\Xi_{12,7} \\
\Psi_{12,39}(X) &= 32\Xi_{12,1} + 384\Xi_{12,3} + 17312\Xi_{12,5} \\ &\quad +3264\Xi_{12,7}
\end{aligned}$$

coset weight 13

$$\Xi_{13,1} = X^{13} - 104X^{23} + 1442X^{25} - 2732X^{27} + 4379X^{29} - 938X^{31} \cdots$$
$$\Xi_{13,2} = X^{15} + 168X^{23} + 518X^{27} + 570X^{29} + 791X^{31} \cdots$$
$$\Xi_{13,3} = X^{17} - 117X^{23} + 609X^{25} - 1416X^{27} + 1736X^{29} - 813X^{31} \cdots$$
$$\Xi_{13,4} = X^{19} - 37X^{23} + 175X^{25} - 390X^{27} + 469X^{29} - 218X^{31} \cdots$$
$$\Xi_{13,5} = 11X^{21} + 3549X^{23} - 12047X^{25} + 32599X^{27} - 30450X^{29} + 22722X^{31} \cdots$$
$$\Xi_{13,6} = 455X^{23} - 1547X^{25} + 4158X^{27} - 3902X^{29} + 2884X^{31} \cdots$$

$$\Psi_{13,1}(X) = \Xi_{13,1} + 59\Xi_{13,2} + 675\Xi_{13,3} + 4025\Xi_{13,4} + 1692\Xi_{13,5} - 12572\Xi_{13,6}$$
$$\Psi_{13,2}(X) = \Xi_{13,1} + 4057\Xi_{13,4} + 1692\Xi_{13,5} - 12574\Xi_{13,6} + 61\Xi_{13,2} + 661\Xi_{13,3}$$
$$\Psi_{13,3}(X) = 2\Xi_{13,1} + 78\Xi_{13,2} + 622\Xi_{13,3} + 4002\Xi_{13,4} + 1704\Xi_{13,5} - 12688\Xi_{13,6}$$
$$\Psi_{13,4}(X) = 3\Xi_{13,1} + 71\Xi_{13,2} + 639\Xi_{13,3} + 3995\Xi_{13,4} + 1700\Xi_{13,5} - 12650\Xi_{13,6}$$
$$\Psi_{13,5}(X) = 3\Xi_{13,1} + 73\Xi_{13,2} + 625\Xi_{13,3} + 4027\Xi_{13,4} + 1700\Xi_{13,5} - 12652\Xi_{13,6}$$
$$\Psi_{13,6}(X) = 4\Xi_{13,1} + 60\Xi_{13,2} + 684\Xi_{13,3} + 3924\Xi_{13,4} + 1696\Xi_{13,5} - 12608\Xi_{13,6}$$
$$\Psi_{13,7}(X) = 4\Xi_{13,1} + 68\Xi_{13,2} + 628\Xi_{13,3} + 4052\Xi_{13,4} + 1696\Xi_{13,5} - 12616\Xi_{13,6}$$
$$\Psi_{13,8}(X) = 4\Xi_{13,1} + 92\Xi_{13,2} + 572\Xi_{13,3} + 4004\Xi_{13,4} + 1712\Xi_{13,5} - 12768\Xi_{13,6}$$
$$\Psi_{13,9}(X) = 4\Xi_{13,1} + 60\Xi_{13,2} + 700\Xi_{13,3} + 3780\Xi_{13,4} + 1744\Xi_{13,5} - 12992\Xi_{13,6}$$
$$\Psi_{13,10}(X) = 5\Xi_{13,1} + 61\Xi_{13,2} + 645\Xi_{13,3} + 4045\Xi_{13,4} + 1692\Xi_{13,5} - 12578\Xi_{13,6}$$
$$\Psi_{13,11}(X) = 6\Xi_{13,1} + 74\Xi_{13,2} + 634\Xi_{13,3} + 3926\Xi_{13,4} + 1704\Xi_{13,5} - 12688\Xi_{13,6}$$
$$\Psi_{13,12}(X) = 6\Xi_{13,1} + 76\Xi_{13,2} + 620\Xi_{13,3} + 3958\Xi_{13,4} + 1704\Xi_{13,5} - 12690\Xi_{13,6}$$
$$\Psi_{13,13}(X) = 6\Xi_{13,1} + 82\Xi_{13,2} + 562\Xi_{13,3} + 4198\Xi_{13,4} + 1656\Xi_{13,5} - 12312\Xi_{13,6}$$
$$\Psi_{13,14}(X) = 7\Xi_{13,1} + 69\Xi_{13,2} + 637\Xi_{13,3} + 3951\Xi_{13,4} + 1700\Xi_{13,5} - 12652\Xi_{13,6}$$
$$\Psi_{13,15}(X) = 7\Xi_{13,1} + 71\Xi_{13,2} + 623\Xi_{13,3} + 3983\Xi_{13,4} + 1700\Xi_{13,5} - 12654\Xi_{13,6}$$
$$\Psi_{13,16}(X) = 8\Xi_{13,1} + 64\Xi_{13,2} + 640\Xi_{13,3} + 3976\Xi_{13,4} + 1696\Xi_{13,5} - 12616\Xi_{13,6}$$
$$\Psi_{13,17}(X) = 8\Xi_{13,1} + 72\Xi_{13,2} + 584\Xi_{13,3} + 4104\Xi_{13,4} + 1696\Xi_{13,5} - 12624\Xi_{13,6}$$
$$\Psi_{13,18}(X) = 10\Xi_{13,1} + 70\Xi_{13,2} + 646\Xi_{13,3} + 3850\Xi_{13,4} + 1704\Xi_{13,5} - 12688\Xi_{13,6}$$
$$\Psi_{13,19}(X) = 10\Xi_{13,1} + 78\Xi_{13,2} + 590\Xi_{13,3} + 3978\Xi_{13,4} + 1704\Xi_{13,5} - 12696\Xi_{13,6}$$
$$\Psi_{13,20}(X) = 11\Xi_{13,1} + 71\Xi_{13,2} + 607\Xi_{13,3} + 3971\Xi_{13,4} + 1700\Xi_{13,5} - 12658\Xi_{13,6}$$
$$\Psi_{13,21}(X) = 12\Xi_{13,1} + 52\Xi_{13,2} + 596\Xi_{13,3} + 4204\Xi_{13,4} + 1680\Xi_{13,5} - 12480\Xi_{13,6}$$
$$\Psi_{13,22}(X) = 12\Xi_{13,1} + 60\Xi_{13,2} + 668\Xi_{13,3} + 3756\Xi_{13,4} + 1744\Xi_{13,5} - 13000\Xi_{13,6}$$
$$\Psi_{13,23}(X) = 12\Xi_{13,1} + 92\Xi_{13,2} + 540\Xi_{13,3} + 3980\Xi_{13,4} + 1712\Xi_{13,5} - 12776\Xi_{13,6}$$
$$\Psi_{13,24}(X) = 14\Xi_{13,1} + 50\Xi_{13,2} + 882\Xi_{13,3} + 2254\Xi_{13,4} + 2296\Xi_{13,5} - 17408\Xi_{13,6}$$
$$\Psi_{13,25}(X) = 14\Xi_{13,1} + 66\Xi_{13,2} + 642\Xi_{13,3} + 3918\Xi_{13,4} + 1656\Xi_{13,5} - 12304\Xi_{13,6}$$
$$\Psi_{13,26}(X) = 16\Xi_{13,1} + 40\Xi_{13,2} + 664\Xi_{13,3} + 4000\Xi_{13,4} + 1680\Xi_{13,5} - 12472\Xi_{13,6}$$
$$\Psi_{13,27}(X) = 16\Xi_{13,1} + 96\Xi_{13,2} + 512\Xi_{13,3} + 3888\Xi_{13,4} + 1760\Xi_{13,5} - 13168\Xi_{13,6}$$
$$\Psi_{13,28}(X) = 24\Xi_{13,1} + 72\Xi_{13,2} + 520\Xi_{13,3} + 4056\Xi_{13,4} + 1696\Xi_{13,5} - 12640\Xi_{13,6}$$

coset weight 14

$$\Xi_{14,1} = X^{14} - 14X^{24} + 1750X^{26} - 5192X^{28} + 11305X^{30} - 12628X^{32} + \cdots$$
$$\Xi_{14,2} = X^{16} - 122X^{24} + 1312X^{26} - 3912X^{28} + 7776X^{30} - 9086X^{32} + \cdots$$
$$\Xi_{14,3} = X^{18} - 7X^{24} + 266X^{26} - 724X^{28} + 1589X^{30} - 1738X^{32} + \cdots$$
$$\Xi_{14,4} = X^{20} - 46X^{24} + 256X^{26} - 721X^{28} + 1280X^{30} - 1540X^{32} + \cdots$$
$$\Xi_{14,5} = X^{22} + 695534X^{24} - 3252963X^{26} + 9862856X^{28} - 16607262X^{30} + 20951700X^{32} - \cdots$$
$$\Xi_{14,6} = 455X^{24} - 2128X^{26} + 6452X^{28} - 10864X^{30} + 13706X^{32} - \cdots$$

$$\begin{aligned}
\Psi_{14,1} &= 4\Xi_{14,1}+216\Xi_{14,2}+1828\Xi_{14,3}\\ &\quad +8768\Xi_{14,4}+36368\Xi_{14,5}-55592580\Xi_{14,6}\\
\Psi_{14,2} &= 6\Xi_{14,1}+236\Xi_{14,2}+1718\Xi_{14,3}\\ &\quad +8928\Xi_{14,4}+36376\Xi_{14,5}-55604790\Xi_{14,6}\\
\Psi_{14,3} &= 8\Xi_{14,1}+224\Xi_{14,2}+1736\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36256\Xi_{14,5}-55421352\Xi_{14,6}\\
\Psi_{14,4} &= 12\Xi_{14,1}+224\Xi_{14,2}+1708\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36336\Xi_{14,5}-55543644\Xi_{14,6}\\
\Psi_{14,5} &= 16\Xi_{14,1}+224\Xi_{14,2}+1680\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36416\Xi_{14,5}-55665936\Xi_{14,6}\\
\Psi_{14,6} &= 16\Xi_{14,1}+256\Xi_{14,2}+1552\Xi_{14,3}\\ &\quad +9088\Xi_{14,4}+36544\Xi_{14,5}-55861584\Xi_{14,6}\\
\Psi_{14,7} &= 20\Xi_{14,1}+200\Xi_{14,2}+1716\Xi_{14,3}\\ &\quad +9024\Xi_{14,4}+36176\Xi_{14,5}-55299060\Xi_{14,6}\\
\Psi_{14,8} &= 20\Xi_{14,1}+224\Xi_{14,2}+1652\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36496\Xi_{14,5}-55788228\Xi_{14,6}\\
\Psi_{14,9} &= 20\Xi_{14,1}+232\Xi_{14,2}+1588\Xi_{14,3}\\ &\quad +9152\Xi_{14,4}+36304\Xi_{14,5}-55494708\Xi_{14,6}\\
\Psi_{14,10} &= 24\Xi_{14,1}+176\Xi_{14,2}+1944\Xi_{14,3}\\ &\quad +7808\Xi_{14,4}+39584\Xi_{14,5}-60508824\Xi_{14,6}\\
\Psi_{14,11} &= 24\Xi_{14,1}+192\Xi_{14,2}+1752\Xi_{14,3}\\ &\quad +8832\Xi_{14,4}+36448\Xi_{14,5}-55714872\Xi_{14,6}\\
\Psi_{14,12} &= 24\Xi_{14,1}+208\Xi_{14,2}+1624\Xi_{14,3}\\ &\quad +9216\Xi_{14,4}+36064\Xi_{14,5}-55127832\Xi_{14,6}\\
\Psi_{14,13} &= 24\Xi_{14,1}+224\Xi_{14,2}+1624\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36576\Xi_{14,5}-55910520\Xi_{14,6}\\
\Psi_{14,14} &= 26\Xi_{14,1}+212\Xi_{14,2}+1642\Xi_{14,3}\\ &\quad +8992\Xi_{14,4}+36456\Xi_{14,5}-55727082\Xi_{14,6}\\
\Psi_{14,15} &= 32\Xi_{14,1}+224\Xi_{14,2}+1568\Xi_{14,3}\\ &\quad +8960\Xi_{14,4}+36736\Xi_{14,5}-56155104\Xi_{14,6}\\
\Psi_{14,16} &= 36\Xi_{14,1}+200\Xi_{14,2}+1604\Xi_{14,3}\\ &\quad +9024\Xi_{14,4}+36496\Xi_{14,5}-55788228\Xi_{14,6}\\
\Psi_{14,17} &= 40\Xi_{14,1}+208\Xi_{14,2}+1512\Xi_{14,3}\\ &\quad +9216\Xi_{14,4}+36384\Xi_{14,5}-55617000\Xi_{14,6}\\
\Psi_{14,18} &= 48\Xi_{14,1}+128\Xi_{14,2}+1712\Xi_{14,3}\\ &\quad +9216\Xi_{14,4}+35776\Xi_{14,5}-54687600\Xi_{14,6}\\
\Psi_{14,19} &= 48\Xi_{14,1}+128\Xi_{14,2}+1904\Xi_{14,3}\\ &\quad +7936\Xi_{14,4}+39424\Xi_{14,5}-60264240\Xi_{14,6}\\
\Psi_{14,20} &= 64\Xi_{14,1}+3136\Xi_{14,3}+73472\Xi_{14,5}\\ &\quad -112312640\Xi_{14,6}\\
\Psi_{14,21} &= 64\Xi_{14,1}+64\Xi_{14,2}+2112\Xi_{14,3}\\ &\quad +7680\Xi_{14,4}+37632\Xi_{14,5}-57524928\Xi_{14,6}\\
\Psi_{14,22} &= 64\Xi_{14,1}+160\Xi_{14,2}+1472\Xi_{14,3}\\ &\quad +9344\Xi_{14,4}+36224\Xi_{14,5}-55372416\Xi_{14,6}
\end{aligned}$$

coset weight 15

$$\begin{aligned}
\Psi_{15,1}(X) &= 16X^{15}+768X^{17}+4240X^{19}+17776X^{21}\\ &\quad +67536X^{23}+182000X^{25}\\ &\quad +383824X^{27}+635568X^{29}+805424X^{31}+\cdots\\
\Psi_{15,2}(X) &= 40X^{15}+728X^{17}+4048X^{19}+18480X^{21}\\ &\quad +66608X^{23}+182224X^{25}\\ &\quad +385424X^{27}+632432X^{29}+807168X^{31}+\cdots\\
\Psi_{15,3}(X) &= 66X^{15}+658X^{17}+4032X^{19}+18656X^{21}\\ &\quad +66472X^{23}+182280X^{25}\\ &\quad +385504X^{27}+631936X^{29}+807548X^{31}+\cdots\\
\Psi_{15,4}(X) &= 72X^{15}+600X^{17}+4272X^{19}+18128X^{21}\\ &\quad +67024X^{23}+182448X^{25}\\ &\quad +384112X^{27}+633872X^{29}+806624X^{31}+\cdots\\
\Psi_{15,5}(X) &= 72X^{15}+616X^{17}+4128X^{19}+18656X^{21}\\ &\quad +66112X^{23}+182784X^{25}\\ &\quad +385696X^{27}+630880X^{29}+808208X^{31}+\cdots\\
\Psi_{15,6}(X) &= 96X^{15}+576X^{17}+3936X^{19}+19360X^{21}\\ &\quad +65184X^{23}+183008X^{25}\\ &\quad +387296X^{27}+627744X^{29}+809952X^{31}+\cdots\\
\Psi_{15,7}(X) &= 112X^{15}+336X^{17}+5600X^{19}+12320X^{21}\\ &\quad +85344X^{23}+142240X^{25}\\ &\quad +445536X^{27}+572832X^{29}+832832X^{31}+\cdots
\end{aligned}$$

coset weight 16

$$\begin{aligned}
\Psi_{16,1}(X) &= 64X^{16}+2560X^{18}+7680X^{20}+36352X^{22}\\ &\quad +116480X^{24}+268800X^{26}\\ &\quad +512512X^{28}+740864X^{30}+823680X^{32}+\cdots\\
\Psi_{16,2}(X) &= 224X^{16}+1792X^{18}+8960X^{20}+36096X^{22}\\ &\quad +113792X^{24}+274176X^{26}\\ &\quad +509184X^{28}+736512X^{30}+832832X^{32}+\cdots\\
\Psi_{16,3}(X) &= 272X^{16}+1472X^{18}+10112X^{20}+33088X^{22}\\ &\quad +119232X^{24}+268736X^{26}\\ &\quad +509056X^{28}+745280X^{30}+819808X^{32}+\cdots\\
\Psi_{16,4}(X) &= 384X^{16}+1024X^{18}+10240X^{20}+35840X^{22}\\ &\quad +111104X^{24}+279552X^{26}\\ &\quad +505856X^{28}+732160X^{30}+841984X^{32}+\cdots\\
\Psi_{16,5}(X) &= 448X^{16}+17920X^{20}+227584X^{24}\\ &\quad +1018368X^{28}+1665664X^{32}+\cdots
\end{aligned}$$

coset weight 17

$$\begin{aligned}
\Psi_{17,1}(X) &= 1008X^{17}+2576X^{19}+24816X^{21}+47376X^{23}\\ &\quad +222768X^{25}+325584X^{27}\\ &\quad +690480X^{29}+782544X^{31}+\cdots
\end{aligned}$$

coset weight 18

$$\begin{aligned}
\Psi_{18,1}(X) &= 3584X^{18}+72192X^{22}+548352X^{26}\\ &\quad +1473024X^{30}+\cdots
\end{aligned}$$

Table 4-1. Cosets, Their Destinations, and Multiplicities

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| 0 | 1 | 1 |
| $\Psi_1$ | $\Phi_1$ | 64 |
| $\Psi_2$ | $\Phi_2$ | 2016 |
| $\Psi_3$ | $\Phi_3$ | 41664 |
| $\Psi_{4,1}$ | $\Phi_{4,1}$ | 624960 |
| $\Psi_{4,2}$ | $\Phi_{4,2}$ | 10416 |
| $\Psi_{5,1}$ | $\Phi_5$ | 6999552 |
| $\Psi_{5,2}$ | $\Phi_3$ | 624960 |
| $\Psi_{6,1}$ | $\Phi_{6,1}$ | 55996416 |
| $\Psi_{6,2}$ | $\Phi_{6,2}$ | 1166592 |
| $\Psi_{6,3}$ | $\Phi_{4,1}$ | 17498880 |
| $\Psi_{6,4}$ | $\Phi_2$ | 312480 |
| $\Psi_{7,1}$ | $\Phi_7$ | 255983616 |
| $\Psi_{7,2}$ | $\Phi_5$ | 55996416 |
| $\Psi_{7,3}$ | $\Phi_5$ | 279982080 |
| $\Psi_{7,4}$ | $\Phi_5$ | 11665920 |
| $\Psi_{7,5}$ | $\Phi_3$ | 17498880 |
| $\Psi_{7,6}$ | $\Phi_1$ | 89280 |
| $\Psi_{8,1}$ | $\Phi_8$ | 31997952 |
| $\Psi_{8,2}$ | $\Phi_{6,1}$ | 895942656 |
| $\Psi_{8,3}$ | $\Phi_{6,1}$ | 2239856640 |
| $\Psi_{8,4}$ | $\Phi_{4,1}$ | 209986560 |
| $\Psi_{8,5}$ | $\Phi_{6,2}$ | 34997760 |
| $\Psi_{8,6}$ | $\Phi_{6,1}$ | 559964160 |
| $\Psi_{8,7}$ | $\Phi_{4,1}$ | 419973120 |
| $\Psi_{8,8}$ | $\Phi_{4,1}$ | 13124160 |
| $\Psi_{8,9}$ | $\Phi_{4,2}$ | 1093680 |
| $\Psi_{8,10}$ | $\Phi_2$ | 2499840 |
| $\Psi_{8,11}$ | $\Phi_0$ | 1395 |

Table 4-2.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{9,1}$ | $\Phi_7$ | 995491840 |
| $\Psi_{9,2}$ | $\Phi_7$ | 1791885312 |
| $\Psi_{9,3}$ | $\Phi_5$ | 6719569920 |
| $\Psi_{9,4}$ | $\Phi_5$ | 1119928320 |
| $\Psi_{9,5}$ | $\Phi_7$ | 8959426560 |
| $\Psi_{9,6}$ | $\Phi_5$ | 839946240 |
| $\Psi_{9,7}$ | $\Phi_5$ | 4479713280 |
| $\Psi_{9,8}$ | $\Phi_3$ | 279982080 |
| $\Psi_{9,9}$ | $\Phi_5$ | 839946240 |
| $\Psi_{9,10}$ | $\Phi_5$ | 629959680 |
| $\Psi_{9,11}$ | $\Phi_3$ | 52496640 |
| $\Psi_{9,12}$ | $\Phi_3$ | 39997440 |
| $\Psi_{10,1}$ | $\Phi_{6,1}$ | 6719569920 |
| $\Psi_{10,2}$ | $\Phi_{4,1}$ | 5039677440 |
| $\Psi_{10,3}$ | $\Phi_{6,1}$ | 26878279680 |
| $\Psi_{10,4}$ | $\Phi_{6,1}$ | 8959426560 |
| $\Psi_{10,5}$ | $\Phi_8$ | 9955491840 |
| $\Psi_{10,6}$ | $\Phi_{6,1}$ | 26878279680 |
| $\Psi_{10,7}$ | $\Phi_{6,2}$ | 1119928320 |
| $\Psi_{10,8}$ | $\Phi_{6,1}$ | 6719569920 |
| $\Psi_{10,9}$ | $\Phi_{4,1}$ | 8959426560 |
| $\Psi_{10,10}$ | $\Phi_{4,1}$ | 1119928320 |
| $\Psi_{10,11}$ | $\Phi_{6,1}$ | 26878279680 |
| $\Psi_{10,12}$ | $\Phi_{6,1}$ | 335978496 |
| $\Psi_{10,13}$ | $\Phi_{6,1}$ | 10079354880 |
| $\Psi_{10,14}$ | $\Phi_{4,1}$ | 1259919360 |
| $\Psi_{10,15}$ | $\Phi_{4,1}$ | 839946240 |
| $\Psi_{10,16}$ | $\Phi_2$ | 52496640 |
| $\Psi_{10,17}$ | $\Phi_{6,1}$ | 839946240 |
| $\Psi_{10,18}$ | $\Phi_{6,2}$ | 52496640 |
| $\Psi_{10,19}$ | $\Phi_{4,1}$ | 319979520 |
| $\Psi_{10,20}$ | $\Phi_{4,2}$ | 9999360 |
| $\Psi_{10,21}$ | $\Phi_{4,1}$ | 157489920 |
| $\Psi_{10,22}$ | $\Phi_{4,1}$ | 69995520 |
| $\Psi_{10,23}$ | $\Phi_2$ | 1749888 |

Table 4-3.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{11,1}$ | $\Phi_5$ | 6719569920 |
| $\Psi_{11,2}$ | $\Phi_7$ | 53756559360 |
| $\Psi_{11,3}$ | $\Phi_5$ | 80634839040 |
| $\Psi_{11,4}$ | $\Phi_7$ | 26878279680 |
| $\Psi_{11,5}$ | $\Phi_5$ | 6719569920 |
| $\Psi_{11,6}$ | $\Phi_3$ | 6719569920 |
| $\Psi_{11,7}$ | $\Phi_5$ | 80634839040 |
| $\Psi_{11,8}$ | $\Phi_7$ | 10751311872 |
| $\Psi_{11,9}$ | $\Phi_7$ | 80634839040 |
| $\Psi_{11,10}$ | $\Phi_5$ | 20158709760 |
| $\Psi_{11,11}$ | $\Phi_5$ | 10079354880 |
| $\Psi_{11,12}$ | $\Phi_5$ | 26878279680 |
| $\Psi_{11,13}$ | $\Phi_7$ | 26878279680 |
| $\Psi_{11,14}$ | $\Phi_5$ | 40317419520 |
| $\Psi_{11,15}$ | $\Phi_3$ | 1679892480 |

Table 4-3 (continued).

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{11,16}$ | $\Phi_5$ | 4479713280 |
| $\Psi_{11,17}$ | $\Phi_7$ | 26878279680 |
| $\Psi_{11,18}$ | $\Phi_5$ | 1679892480 |
| $\Psi_{11,19}$ | $\Phi_5$ | 5039677440 |
| $\Psi_{11,20}$ | $\Phi_5$ | 209986560 |
| $\Psi_{11,21}$ | $\Phi_3$ | 319979520 |
| $\Psi_{11,22}$ | $\Phi_5$ | 2239856640 |
| $\Psi_{11,23}$ | $\Phi_5$ | 671956992 |
| $\Psi_{11,24}$ | $\Phi_3$ | 209986560 |
| $\Psi_{11,25}$ | $\Phi_3$ | 55996416 |
| $\Psi_{11,26}$ | $\Phi_1$ | 3499776 |

Table 4-4.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{12,1}$ | $\Phi_{4,1}$ | 26878279680 |
| $\Psi_{12,2}$ | $\Phi_{6,1}$ | 80634839040 |
| $\Psi_{12,3}$ | $\Phi_{6,1}$ | 107513118720 |
| $\Psi_{12,4}$ | $\Phi_8$ | 5375655936 |
| $\Psi_{12,5}$ | $\Phi_{6,1}$ | 80634839040 |
| $\Psi_{12,6}$ | $\Phi_{6,1}$ | 215026237440 |
| $\Psi_{12,7}$ | $\Phi_{6,1}$ | 40317419520 |
| $\Psi_{12,8}$ | $\Phi_{6,2}$ | 10079354880 |
| $\Psi_{12,8}$ | $\Phi_{6,1}$ | 181180661760 |
| $\Psi_{12,8}$ | $\Phi_{4,1}$ | 80510976000 |
| $\Psi_{12,9}$ | $\Phi_{4,1}$ | 3359784960 |
| $\Psi_{12,10}$ | $\Phi_2$ | 559964160 |
| $\Psi_{12,11}$ | $\Phi_{4,1}$ | 20158709760 |
| $\Psi_{12,12}$ | $\Phi_{6,1}$ | 80510976000 |
| $\Psi_{12,13}$ | $\Phi_{4,1}$ | 10079354880 |
| $\Psi_{12,14}$ | $\Phi_{6,1}$ | 160939376640 |
| $\Psi_{12,15}$ | $\Phi_{6,2}$ | 139991040 |
| $\Psi_{12,16}$ | $\Phi_{4,1}$ | 10079354880 |
| $\Psi_{12,17}$ | $\Phi_{6,1}$ | 7091159040 |
| $\Psi_{12,18}$ | $\Phi_{6,1}$ | 80634839040 |
| $\Psi_{12,19}$ | $\Phi_8$ | 3359784960 |
| $\Psi_{12,20}$ | $\Phi_{4,2}$ | 209986560 |
| $\Psi_{12,21}$ | $\Phi_{4,1}$ | 6719569920 |
| $\Psi_{12,21}$ | $\Phi_{4,1}$ | 209986560 |
| $\Psi_{12,22}$ | $\Phi_{6,1}$ | 20127744000 |
| $\Psi_{12,23}$ | $\Phi_{6,1}$ | 13439139840 |
| $\Psi_{12,24}$ | $\Phi_2$ | 159989760 |
| $\Psi_{12,25}$ | $\Phi_{6,1}$ | 21998075904 |
| $\Psi_{12,26}$ | $\Phi_{6,1}$ | 6719569920 |
| $\Psi_{12,27}$ | $\Phi_{4,1}$ | 3318497280 |
| $\Psi_{12,28}$ | $\Phi_{4,1}$ | 5019033600 |
| $\Psi_{12,29}$ | $\Phi_{6,1}$ | 20158709760 |
| $\Psi_{12,30}$ | $\Phi_{4,1}$ | 6843432960 |
| $\Psi_{12,31}$ | $\Phi_{4,1}$ | 629959680 |
| $\Psi_{12,32}$ | $\Phi_{6,2}$ | 314979840 |
| $\Psi_{12,33}$ | $\Phi_{4,1}$ | 26248320 |
| $\Psi_{12,34}$ | $\Phi_2$ | 55996416 |
| $\Psi_{12,35}$ | $\Phi_{4,1}$ | 419973120 |
| $\Psi_{12,36}$ | $\Phi_{6,2}$ | 20998656 |
| $\Psi_{12,37}$ | $\Phi_{4,2}$ | 6562080 |
| $\Psi_{12,38}$ | $\Phi_{4,2}$ | 874944 |
| $\Psi_{12,39}$ | $\Phi_0$ | 54684 |

Table 4-5.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{13,1}$ | $\Phi_5$ | 107513118720 |
| $\Psi_{13,2}$ | $\Phi_7$ | 53756559360 |
| $\Psi_{13,3}$ | $\Phi_5$ | 80634839040 |
| $\Psi_{13,4}$ | $\Phi_7$ | 161269678080 |
| $\Psi_{13,5}$ | $\Phi_5$ | 215026237440 |
| $\Psi_{13,6}$ | $\Phi_5$ | 2239856640 |
| $\Psi_{13,7}$ | $\Phi_5$ | 40317419520 |
| $\Psi_{13,8}$ | $\Phi_5$ | 13439139840 |
| $\Psi_{13,9}$ | $\Phi_3$ | 13439139840 |
| $\Psi_{13,10}$ | $\Phi_7$ | 53756559360 |
| $\Psi_{13,11}$ | $\Phi_5$ | 80634839040 |
| $\Psi_{13,12}$ | $\Phi_7$ | 215026237440 |
| $\Psi_{13,13}$ | $\Phi_3$ | 6719569920 |
| $\Psi_{13,14}$ | $\Phi_5$ | 107513118720 |
| $\Psi_{13,15}$ | $\Phi_7$ | 115192627200 |
| $\Psi_{13,16}$ | $\Phi_5$ | 161269678080 |
| $\Psi_{13,17}$ | $\Phi_5$ | 5039677440 |
| $\Psi_{13,18}$ | $\Phi_5$ | 13439139840 |
| $\Psi_{13,19}$ | $\Phi_5$ | 80634839040 |
| $\Psi_{13,20}$ | $\Phi_7$ | 64507871232 |
| $\Psi_{13,21}$ | $\Phi_5$ | 839946240 |
| $\Psi_{13,22}$ | $\Phi_3$ | 5039677440 |
| $\Psi_{13,23}$ | $\Phi_5$ | 5039677440 |
| $\Psi_{13,24}$ | $\Phi_1$ | 22855680 |
| $\Psi_{13,25}$ | $\Phi_3$ | 1119928320 |
| $\Psi_{13,26}$ | $\Phi_5$ | 671956992 |
| $\Psi_{13,27}$ | $\Phi_3$ | 209986560 |
| $\Psi_{13,28}$ | $\Phi_5$ | 839946240 |

Table 4-6.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{14,1}$ | $\Phi_{4,1}$ | 6719569920 |
| $\Psi_{14,2}$ | $\Phi_{6,1}$ | 26878279680 |
| $\Psi_{14,3}$ | $\Phi_8$ | 959938560 |
| $\Psi_{14,4}$ | $\Phi_{6,1}$ | 107513118720 |
| $\Psi_{14,5}$ | $\Phi_{4,1}$ | 67195699200 |
| $\Psi_{14,5}$ | $\Phi_{4,2}$ | 419973120 |
| $\Psi_{14,5}$ | $\Phi_{6,1}$ | 161269678080 |
| $\Psi_{14,5}$ | $\Phi_{6,2}$ | 6719569920 |
| $\Psi_{14,6}$ | $\Phi_{4,1}$ | 5039677440 |
| $\Psi_{14,7}$ | $\Phi_{6,1}$ | 10079354880 |
| $\Psi_{14,8}$ | $\Phi_{6,1}$ | 174708817920 |
| $\Psi_{14,9}$ | $\Phi_{4,1}$ | 20158709760 |
| $\Psi_{14,10}$ | $\Phi_2$ | 839946240 |
| $\Psi_{14,11}$ | $\Phi_{4,1}$ | 31917957120 |
| $\Psi_{14,12}$ | $\Phi_{4,1}$ | 1259919360 |
| $\Psi_{14,13}$ | $\Phi_8$ | 3839754240 |
| $\Psi_{14,14}$ | $\Phi_{6,1}$ | 112888774656 |
| $\Psi_{14,15}$ | $\Phi_{6,2}$ | 419973120 |
| $\Psi_{14,16}$ | $\Phi_{6,1}$ | 6719569920 |
| $\Psi_{14,17}$ | $\Phi_{4,1}$ | 2519838720 |
| $\Psi_{14,18}$ | $\Phi_{6,2}$ | 104993280 |
| $\Psi_{14,19}$ | $\Phi_2$ | 104993280 |
| $\Psi_{14,20}$ | $\Phi_0$ | 357120 |
| $\Psi_{14,21}$ | $\Phi_{4,2}$ | 17498880 |
| $\Psi_{14,22}$ | $\Phi_{4,1}$ | 104993280 |

Table 4-7.

| $W_U(X)$ | $W_{\varphi(U)}(X)$ | multiplicity |
|---|---|---|
| $\Psi_{15,1}$ | $\Phi_3$ | 223985664 |
| $\Psi_{15,2}$ | $\Phi_5$ | 3359784960 |
| $\Psi_{15,3}$ | $\Phi_7$ | 30718033920 |
| $\Psi_{15,4}$ | $\Phi_3$ | 6719569920 |
| $\Psi_{15,5}$ | $\Phi_5$ | 3359784960 |
| $\Psi_{15,6}$ | $\Phi_3$ | 839946240 |
| $\Psi_{15,7}$ | $\Phi_1$ | 29998080 |
| $\Psi_{16,1}$ | $\Phi_{4,2}$ | 3499776 |
| $\Psi_{16,2}$ | $\Phi_{6,2}$ | 104993280 |
| $\Psi_{16,3}$ | $\Phi_2$ | 335978496 |
| $\Psi_{16,4}$ | $\Phi_{4,2}$ | 13124160 |
| $\Psi_{16,5}$ | $\Phi_0$ | 468720 |
| $\Psi_{17}$ | $\Phi_1$ | 10665984 |
| $\Psi_{18}$ | $\Phi_0$ | 166656 |

## 9. Appendix

We give instances of the coset leaders in $RM(2,6)$ which lead to the identical coset weight enumerator in $RM(2,6)$ but lead to different coset weight enumerators in $RM(3,6)$. In the Overview we named those cosets as optical isomeric cosets. Note that the shapes of the coset leaders may depend on the choice of the generator matrix of the code $RM(2,6)$. We employ the generator matrix obtained after the command done for MAGMA in the section 5.

The three vectors

$$\begin{aligned} \mathbf{v}_1 &= (00000101101000100000000010000000 \\ &\quad 00100000000000000100001000010000) \\ \mathbf{v}_2 &= (00110000011000100000000010000000 \\ &\quad 00100000000000000100001000010000) \\ \mathbf{v}_3 &= (00100110010000100000000010000000 \\ &\quad 00100000000000000100001000010000) \end{aligned}$$

are the coset leaders of weight 10 in $RM(2,6)$ and have the same coset weight enumerator $\Psi_{10,7}(X)$. But in $RM(3,6)$ $\mathbf{v}_1$ (resp. $\mathbf{v}_2$, $\mathbf{v}_3$) leads to a coset with the coset weight enumerator $\Phi_{4,1}(X)$ (resp. $\Phi_{6,1}(X)$, $\Phi_{6,2}(X)$) that are given in the Section 5.

The four vectors

$$\begin{aligned} \mathbf{v}_4 &= (10010001000100101000010010000000 \\ &\quad 01001000100000000000100010001000) \\ \mathbf{v}_5 &= (10010001001000101000010010000000 \\ &\quad 01001000100000000000100010001000) \\ \mathbf{v}_6 &= (11000001000100101000010010000000 \\ &\quad 01001000100000000000100010001000) \\ \mathbf{v}_7 &= (01010001000100101000010010000000 \\ &\quad 01001000100000000000100010001000) \end{aligned}$$

are the coset leaders of weight 14 in $RM(2,6)$ and have the same coset weight enumerator $\Psi_{14,5}(X)$. But in $RM(3,6)$

$\mathbf{v}_4$ (resp. $\mathbf{v}_5$, $\mathbf{v}_6$, $\mathbf{v}_7$,) leads to a coset with the coset weight enumerator $\Phi_{4,1}(X)$ (resp. $\Phi_{4,2}(X)$, $\Phi_{6,1}(X)$, $\Phi_{6,2}(X)$).

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