

# Shape derivative of potential energy and energy release rate in fracture mechanics

Masato Kimura and Isao Wakano

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**Abstract.** We study a general mathematical framework for variation of potential energy with respect to domain deformation. It enables rigorous derivation of the integral formulas for the energy release rate in crack problems. Applying a technique of shape sensitivity analysis, we formulate the shape derivative of potential energy as a variational problem with a parameter. Key tools of our abstract theory are a new parameter variational principle and the classical implicit function theorem in Banach spaces.

*Keywords.* shape derivative, variational principle, energy release rate, fracture mechanics

## 1. INTRODUCTION

Many variational problems related to domain deformation have been investigated in the theory of shape derivatives or shape sensitivity analysis [7, 8, 14, 22]. According to the spread of the importance of shape derivatives in various scientific fields, more development of their mathematical foundation has been required. The purpose of this paper is to establish some abstract parameter variational formulas and to illustrate their application to the shape derivative of elastic potential energies. An important example concerns the energy release rate in crack problems, which is known as one of the most fundamental quantities in the theory of fracture mechanics.

Scientific investigation to understand crack evolution process in elastic bodies was originated by Griffith [9] and has been studied from various viewpoints in engineering, physics and mathematics since then. Griffith's contribution to fracture mechanics is even now the fundamental approach to modeling and analyzing crack behavior. We here make reference to only very few extended studies from the mathematical point of view, Cherepanov [4], Rice [21], Ohtsuka [17, 18, 19], Francfort-Marigo [6], Ohtsuka-Khludnev [20], Kovtunen [16], and Bourdin et al. [3] etc. For a more complete list of crack problems in fracture mechanics, see the references cited in the above papers.

In the Griffith's theory and its various extended theories such as [6], the concept of the energy release rate  $G$  plays an important role. According to such theories, we treat crack evolutions in brittle materials with linear elasticity under a quasi-static situation, where the quasi-static situation means that an inertial term of the governing equation is negligible under assumption that velocity of deformation of body is very slow. In the classical Griffith's theory, since it treats a crack before propagation, the quasi-static situ-

ation is assumed and it is widely accepted nowadays. Furthermore, even during crack propagation, the quasi-static assumption is often adopted in many crack propagation modelings, e.g. [6] etc. The elastic energy at a fixed moment is supposed to be given by minimization of an elastostatic energy. According to the Griffith's theory, the surface energy required in the crack evolution is supplied by relaxation of the potential energy along crack growth.

Roughly speaking, the energy release rate  $G$  is defined as follows (see Section 5 for more details). Let  $\Omega_*$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ), which corresponds to the uncracked material under consideration. We assume that a crack  $\Sigma$  exist in  $\Omega_*$ , where  $\Sigma$  is the closure of an  $n - 1$  dimensional hypersurface. The cracked elastic body is represented by  $\Omega_* \setminus \Sigma$ . We consider a virtual crack extension  $\Sigma(t)$  with parameter  $t \in [0, T)$ , where

$$(1) \quad \Sigma = \Sigma(0) \subset \Sigma(t_1) \subset \Sigma(t_2) \quad (0 \leq t_1 \leq t_2 < T).$$

Under the quasi-static assumption, the elastic potential energy  $E(t)$  in  $\Omega(t) := \Omega_* \setminus \Sigma(t)$  is given by

$$(2) \quad E(t) := \min_v \int_{\Omega_* \setminus \Sigma(t)} W(v) dx,$$

where by  $W(v)$  an elastic potential energy density including a body force is denoted, and  $\min_v$  is taken over all possible displacement fields in  $\Omega_* \setminus \Sigma(t)$  with a given boundary condition. For the admissible displacement fields, a given displacement field is imposed only on the part  $\Gamma_D \subset \partial\Omega_*$ . On the other part  $\partial\Omega(t) \setminus \Gamma_D$  including both sides of  $\Sigma(t)$ , the normal stress free condition is imposed for the minimizer on  $\partial\Omega(t)$  implicitly.

The energy release rate  $G$  at  $t = 0$  along the virtual crack extension  $\{\Sigma(t)\}_{0 \leq t < T}$  is given by

$$(3) \quad G := \lim_{t \rightarrow 0} \frac{E(0) - E(t)}{\mathcal{H}^{n-1}(\Sigma(t) \setminus \Sigma)},$$

where  $\mathcal{H}^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure. Since  $E(t) \leq E(0)$ ,  $G \geq 0$  follows if the limit exists. The Griffith's criterion for the brittle crack extension is given by  $G \geq G_c$ , where  $G_c$  is an energy required to create new crack per unit length (area) and it is a constant depending on the material property and the position.

Cherepanov [4] and Rice [21] studied so-called J-integral for straight crack in two dimensional linear elasticity, which is a path-independent integral expressions of the energy release rate. Since these works, theoretical and practical studies of crack evolutions have been much developed by means of such useful mathematical expression of  $G$  in two dimensional case.

As an alternative approach to such energy based arguments, Irwin [12] proposed the notions of fracture toughness and stress intensity factors and he developed arguments based on the singularity of stress fields.

While most of these mathematically rigorous results have been restricted to two dimensional linear elasticity (and often only for straight cracks), Ohtsuka [17], [18], [19], Ohtsuka-Khludnev [20], and Kovtunencko [16] developed mathematical formulations of the energy release rate for general curved cracks in multi-dimensional linear or semi-linear elliptic systems. They proved existence of the energy release rate, and obtained its expression by a domain integral and by a generalized J-integral.

Based on the idea in [18], we shall give a new mathematical framework for shape derivative of potential energy including the energy release rate. In our approach, domain perturbation  $\varphi$  of Lipschitz class is adopted and the shape derivative of minimum potential energy is derived as a Fréchet derivative in a Banach space within an abstract parameter variation formulas, which were first established in [15]. Instead of estimating the limit (3) directly as in [18], [20] and [16], we treat it by means of the Fréchet derivative and the implicit function theorem.

The organization of this paper is as follows. Abstract parameter variation formulas are proved based on the implicit function theorem in Banach spaces in Section 2. In Section 3, a framework of Lipschitz deformation of domains, which includes crack extensions, is introduced. Most of contents in these sections already appeared in [15]. For the readers' convenience, however, we try to include all proofs which were already stated in [15] with some additional propositions.

We consider inhomogeneous anisotropic elastic tensor field and derive the differentiability of the elastic energy with respect to a Lipschitz domain perturbation. The results obtained there include the results in [18] and [20] under a weaker assumption for regularity of domain perturbation. In [20], they assumed that the domain perturbation  $\varphi$  belongs to  $C^2([0, T], W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^n))$  and derived the domain integral expression, whereas, [16] proved it under a weaker assumption  $\varphi \in C^1([0, T], W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n))$ . Our results give an alternative proof of the results in [16], and we can additionally prove higher order differentiability of the elastic potential energy with respect to the domain defor-

mation (Theorem 4.5 and Theorem 5.1).

## 2. PARAMETER VARIATION FORMULAS

We consider a variational problem with a parameter in an abstract setting. For a real valued functional  $J$  defined on a metric space  $S$ ,  $u_0 \in S$  is called a *global minimizer* of  $J$  in  $S$ , if  $J(u_0) \leq J(u)$  for all  $u \in S$ . If there exists an open set  $\mathcal{O} \subset S$  and  $u_0$  is a global minimizer of  $J$  in  $\mathcal{O}$ ,  $u_0 \in S$  is called a *local minimizer* of  $J$ .

Let  $X$  and  $M$  be real Banach spaces. For open subsets  $\mathcal{U}_0 \subset X$  and  $\mathcal{O}_0 \subset M$ , we consider  $J \in C^1(\mathcal{U}_0 \times \mathcal{O}_0, \mathbb{R})$  and  $u \in C^1(\mathcal{O}_0, \mathcal{U}_0)$ . We assume that  $u(\mu)$  is a local minimizer of  $J(\cdot, \mu)$  in  $\mathcal{U}_0$  for each  $\mu \in \mathcal{O}_0$ , and define  $J_*(\mu) := J(u(\mu), \mu)$  for  $\mu \in \mathcal{O}_0$ . Then we have  $J_* \in C^1(\mathcal{O}_0)$  and

$$(4) \quad \begin{aligned} J'_*(\mu) &= D_\mu[J(u(\mu), \mu)] \\ &= \partial_X J(u(\mu), \mu)[u'(\mu)] + \partial_M J(u(\mu), \mu) \\ &= \partial_M J(u(\mu), \mu), \end{aligned}$$

where  $J'_*$  denotes the Fréchet derivative of  $J_*$  and  $D_\mu$  denotes the Fréchet differential operator with respect to  $\mu \in M$ . The symbols  $\partial_X$  and  $\partial_M$  denote the partial Fréchet derivative operators for  $J(u, \mu)$  with respect to  $u \in X$  and  $\mu \in M$ , respectively. The last equality of (4) follows from  $\partial_X J(u(\mu), \mu) = 0 \in X'$ , where  $X'$  denotes the dual space of  $X$ . The formula

$$(5) \quad J'_*(\mu) = \partial_M J(u(\mu), \mu) \quad (\mu \in \mathcal{O}_0),$$

is a simple but essential equation in this paper.

The following fundamental theorem states that the formula (5) is derived under a weaker assumption for regularity.

**Theorem 2.1.** *Let  $X$  and  $M$  be real Banach spaces. For  $\mathcal{U}_0 \subset X$  and an open subset  $\mathcal{O}_0 \subset M$ , we consider a real valued functional  $J : \mathcal{U}_0 \times \mathcal{O}_0 \rightarrow \mathbb{R}$  and a map  $u : \mathcal{O}_0 \rightarrow \mathcal{U}_0$ . We define  $J_*(\mu) := J(u(\mu), \mu)$  for  $\mu \in \mathcal{O}_0$ . We suppose the following conditions.*

1.  $J \in C^0(\mathcal{U}_0 \times \mathcal{O}_0)$ ,  $J(w, \cdot) \in C^1(\mathcal{O}_0)$  for  $w \in \mathcal{U}_0$ , and  $\partial_M J \in C^0(\mathcal{U}_0 \times \mathcal{O}_0, M')$ .
2.  $u \in C^0(\mathcal{O}_0, X)$  and  $u(\mu)$  is a global minimizer of  $J(\cdot, \mu)$  in  $\mathcal{U}_0$  for each  $\mu \in \mathcal{O}_0$ .

Then  $J_* \in C^1(\mathcal{O}_0)$  and (5) holds.

*Proof.* We fix  $\mu_0 \in \mathcal{O}_0$  and we define  $u_0 := u(\mu_0)$  and

$$r(\mu) := J_*(\mu) - J_*(\mu_0) - \partial_M J(u_0, \mu_0)[\mu - \mu_0] \quad (\mu \in \mathcal{O}_0).$$

Since  $u(\mu)$  is a global minimizer and  $u \in C^0(\mathcal{O}_0, X)$ , if  $\mu$

is close to  $\mu_0$ , we have

$$\begin{aligned} r(\mu) &\leq J(u_0, \mu) - J(u_0, \mu_0) - \partial_M J(u_0, \mu_0)[\mu - \mu_0] \\ &= o(\|\mu - \mu_0\|_M), \\ r(\mu) &\geq J(u(\mu), \mu) - J(u(\mu), \mu_0) - \partial_M J(u_0, \mu_0)[\mu - \mu_0] \\ &= \int_0^1 \{\partial_M J(u(\mu), \mu_0 + s(\mu - \mu_0)) \\ &\quad - \partial_M J(u_0, \mu_0)\}[\mu - \mu_0] ds \\ &= o(\|\mu - \mu_0\|_M). \end{aligned}$$

It follows that  $r(\mu) = o(\|\mu - \mu_0\|_M)$  as  $\mu \rightarrow \mu_0$ , and we obtain the formula (5) and  $J'_* \in C^0(\mathcal{O}_0, M')$ .  $\square$

**Corollary 2.2.** *Under the condition of Theorem 2.1, we assume that  $\mathcal{U}_0$  is open. If  $\partial_M J \in C^k(\mathcal{U}_0 \times \mathcal{O}_0, M')$  and  $u \in C^k(\mathcal{O}_0, X)$ , then  $J_* \in C^{k+1}(\mathcal{O}_0)$ .*

*Proof.* This immediately follows from the formula (5).  $\square$

We apply the implicit function theorem in Banach spaces below. The proof is found in [2] and [10] etc. For two Banach spaces  $X$  and  $Y$ ,  $B(X, Y)$  denotes the Banach space which consists of all bounded linear operators from  $X$  to  $Y$ .

**Theorem 2.3** (Implicit function theorem). *Let  $X, Y, Z$  be real Banach spaces and  $U, V$  be open sets in  $X$  and  $Y$ , respectively. We suppose that  $F : U \times V \rightarrow Z$  and  $(x_0, y_0) \in U \times V$  satisfy the conditions:*

1.  $F(x_0, y_0) = 0$ .
2.  $F \in C^0(U \times V, Z)$ .
3.  $F(x, \cdot) \in C^1(V, Z)$  for  $x \in U$  and  $\partial_Y F$  is continuous at  $(x, y) = (x_0, y_0)$ .
4.  $(\partial_Y F(x_0, y_0))^{-1} \in B(Z, Y)$ .

*Then there exist a convex open neighborhood of  $(x_0, y_0)$ ,  $U_0 \times V_0 \subset U \times V$  and  $f \in C^0(U_0, V_0)$ , such that, for  $(x, y) \in U_0 \times V_0$ ,  $F(x, y) = 0$  if and only if  $y = f(x)$ . Moreover, if  $F \in C^k(U \times V, Z)$  ( $k \in \mathbb{N}$ ), then  $f \in C^k(U_0, V_0)$ .*

From Theorem 2.1 and the implicit function theorem, we get the following theorems.

**Theorem 2.4.** *Let  $X$  and  $M$  be real Banach spaces and  $\mathcal{U}$  and  $\mathcal{O}$  be open subsets of  $X$  and  $M$ , respectively. We consider a real valued functional  $J : \mathcal{U} \times \mathcal{O} \rightarrow \mathbb{R}$  and fix  $\mu_0 \in \mathcal{O}$ . We assume*

1.  $J(\cdot, \mu) \in C^2(\mathcal{U})$  for  $\mu \in \mathcal{O}$  and  $\partial_X J \in C^0(\mathcal{U} \times \mathcal{O}, X')$ .
2.  $u_0 \in \mathcal{U}$  satisfies  $\partial_X J(u_0, \mu_0) = 0$ .
3.  $\partial_X^2 J$  is continuous at  $(w, \mu) = (u_0, \mu_0)$ .
4. There exists  $\alpha > 0$  such that  $\partial_X^2 J(u_0, \mu_0)[w, w] \geq \alpha \|w\|_X^2$  for  $w \in X$ .

*Then there exist a convex open neighborhood of  $(u_0, \mu_0)$ ,  $\mathcal{U}_0 \times \mathcal{O}_0 \subset \mathcal{U} \times \mathcal{O}$  and  $u \in C^0(\mathcal{O}_0, \mathcal{U}_0)$ , such that, for  $\mu \in \mathcal{O}_0$ , the following three conditions are equivalent.*

- a.  $w \in \mathcal{U}_0$  is a local minimizer of  $J(\cdot, \mu)$
- b.  $w \in \mathcal{U}_0$  satisfies  $\partial_X J(w, \mu) = 0$ .
- c.  $w = u(\mu)$ .

*In this case,  $u(\mu)$  is a global minimizer of  $J(\cdot, \mu)$  on  $\mathcal{U}_0$ .*

*Proof.* We define a map  $F := \partial_X J$  from  $\mathcal{U} \times \mathcal{O}$  to  $X'$  and apply Theorem 2.3 at  $(w, \mu) = (u_0, \mu_0)$ . From assumption 4 and the Lax-Milgram theorem,  $\partial_X F(u_0, \mu_0) = \partial_X^2 J(u_0, \mu_0)$  becomes a linear topological isomorphism from  $X$  to  $X'$ . Then, from the implicit function theorem there exist a convex open neighborhood of  $(u_0, \mu_0)$ ,  $\mathcal{U}_0 \times \mathcal{O}_0 \subset \mathcal{U} \times \mathcal{O}$  and  $u \in C^0(\mathcal{O}_0, \mathcal{U}_0)$ , such that, for  $\mu \in \mathcal{O}_0$ ,  $w \in \mathcal{U}_0$  satisfies  $\partial_X J(w, \mu) = 0$  if and only if  $w = u(\mu)$ .

From the continuity of  $\partial_X^2 J$  at  $(u_0, \mu_0)$ , without loss of generality, (after replacing  $\mathcal{U}_0$  and  $\mathcal{O}_0$  with smaller ones if needed) we can assume that

$$(6) \quad \partial_X^2 J(v, \mu)[w, w] \geq \frac{\alpha}{2} \|w\|_X^2 \quad \left( \begin{array}{l} \forall w \in X \\ \forall (v, \mu) \in \mathcal{U}_0 \times \mathcal{O}_0 \end{array} \right).$$

For  $\mu \in \mathcal{O}_0$ , if  $w \in \mathcal{U}_0$  is a local minimizer of  $J(\cdot, \mu)$  in  $\mathcal{U}_0$ , the  $\partial_X J(w, \mu) = 0$  follows. Conversely, if  $w \in \mathcal{U}_0$  satisfies  $\partial_X J(w, \mu) = 0$ ,  $w$  is a local minimizer in  $\mathcal{U}_0$  from the condition (6). It also follows from (6) that  $u(\mu)$  is a global minimizer of  $J(\cdot, \mu)$  in  $\mathcal{U}_0$ .  $\square$

**Theorem 2.5.** *Under the condition of Theorem 2.4, we additionally assume that  $\partial_X J \in C^k(\mathcal{U} \times \mathcal{O}, X')$  for some  $k \in \mathbb{N}$ . Then  $u \in C^k(\mathcal{O}_0, \mathcal{U}_0)$  holds.*

*Proof.* The assertion follows from the implicit function theorem.  $\square$

Under the condition of Theorem 2.4, we define

$$J_*(\mu) := J(u(\mu), \mu) \quad (\mu \in \mathcal{O}_0).$$

As a consequence of Theorem 2.5, a sufficient condition for  $J_* \in C^1(\mathcal{O}_0)$  is  $J \in C^1(\mathcal{U} \times \mathcal{O})$  and  $\partial_X J \in C^1(\mathcal{U} \times \mathcal{O}, X')$ . However, the condition  $\partial_X J \in C^1(\mathcal{U} \times \mathcal{O}, X')$  is not necessary due to Theorem 2.1.

**Theorem 2.6.** *Under the condition of Theorem 2.4, we additionally assume that  $J \in C^k(\mathcal{U} \times \mathcal{O})$  for some  $k \in \mathbb{N}$ , then  $J_* \in C^k(\mathcal{O}_0)$  and it satisfies (5).*

*Proof.* From Theorem 2.1,  $J_* \in C^1(\mathcal{O}_0)$  and (5) immediately follows. Since  $u \in C^{k-1}(\mathcal{O}_0, X)$  follows from Theorem 2.5,  $J_* \in C^k(\mathcal{O}_0)$  is obtained from the formula (5).  $\square$

Let us consider the case  $k = 1$  in Theorem 2.6, where  $J_* \in C^1(\mathcal{O}_0)$  is derived under the conditions  $J \in C^1(\mathcal{U} \times \mathcal{O})$  and  $J(\cdot, \mu) \in C^2(\mathcal{U})$ . In this case,  $u \in C^0(\mathcal{O}_0, \mathcal{U}_0)$  holds from Theorem 2.4 but  $u \notin C^1(\mathcal{O}_0, \mathcal{U}_0)$  in general. In order to obtain  $u \in C^1(\mathcal{O}_0, \mathcal{U}_0)$ , we need to assume  $\partial_X J \in C^1(\mathcal{U} \times \mathcal{O}, X')$  (Theorem 2.5). We have Hölder regularity of  $u$  under the condition of Theorem 2.6 with  $k = 1$ .

**Proposition 2.7.** *Under the condition of Theorem 2.4, we additionally assume that  $J \in C^1(\mathcal{U} \times \mathcal{O})$ , then we have*

$$\|u(\mu) - u_0\|_X = o\left(\|\mu - \mu_0\|_M^{1/2}\right) \text{ as } \|\mu - \mu_0\|_M \rightarrow 0.$$

*Proof.* From the proof of Theorem 2.3, there exists  $C > 0$  such that

$$(7) \quad \|u(\mu) - u_0\|_X \leq C \|\partial_X J(u_0, \mu)\|_{X'} \quad (\mu \in \mathcal{O}_0).$$

Let  $\rho_0 > 0$  with  $\{v \in X; \|v - u_0\|_X \leq \rho_0\} \subset \mathcal{U}_0$ . For  $h \in X$  with  $\|h\|_X = 1$ ,  $\mu \in \mathcal{O}_0$  and  $\rho \in (0, \rho_0]$ , we have

$$J(u_0 + \rho h, \mu) = J(u_0, \mu) + \rho \partial_X J(u_0, \mu)[h] + \rho^2 \int_0^1 (1-s) \partial_X^2 J(u_0 + s\rho h, \mu)[h, h] ds.$$

$$\begin{aligned} & \partial_X J(u_0, \mu)[h] \\ &= \partial_X J(u_0, \mu)[h] - \partial_X J(u_0, \mu_0)[h] \\ &= \left\{ \rho^{-1} (J(u_0 + \rho h, \mu) - J(u_0, \mu)) \right. \\ & \quad \left. - \rho \int_0^1 (1-s) \partial_X^2 J(u_0 + s\rho h, \mu)[h, h] ds \right\} \\ & \quad - \left\{ \rho^{-1} (J(u_0 + \rho h, \mu_0) - J(u_0, \mu_0)) \right. \\ & \quad \left. - \rho \int_0^1 (1-s) \partial_X^2 J(u_0 + s\rho h, \mu_0)[h, h] ds \right\} \\ &= \rho^{-1} \int_0^1 \left\{ \partial_M J(u_0 + \rho h, \mu_0 + t(\mu - \mu_0)) \right. \\ & \quad \left. - \partial_M J(u_0, \mu_0 + t(\mu - \mu_0)) \right\} [\mu - \mu_0] dt \\ & \quad - \rho \int_0^1 (1-s) \left\{ \partial_X^2 J(u_0 + s\rho h, \mu) \right. \\ & \quad \left. - \partial_X^2 J(u_0 + s\rho h, \mu_0) \right\} [h, h] ds \end{aligned}$$

For  $r > 0$ , we define

$$S(r) := \left\{ (w, \lambda) \in X \times M; \begin{array}{l} \|w - u_0\|_X \leq r \\ \|\lambda - \mu_0\|_M \leq r^2 \end{array} \right\},$$

$$\begin{aligned} \omega(r) := & \sup_{(w, \lambda) \in S(r)} \|\partial_M J(w, \lambda) - \partial_M J(u_0, \mu_0)\|_{M'} \\ & + \sup_{(w, \lambda) \in S(r)} \|\partial_X^2 J(w, \lambda) - \partial_X^2 J(u_0, \mu_0)\|_{B_2(X, \mathbb{R})}. \end{aligned}$$

We remark that  $\omega(r) \rightarrow 0$  as  $r \rightarrow +0$ . Choosing  $\rho := \|\mu - \mu_0\|_M^{1/2}$ , we obtain

$$\|\partial_X J(u_0, \mu)\|_{X'} \leq 2\omega(\rho)\rho \quad (\mu \in \mathcal{O}_0, \|\mu - \mu_0\|_M \leq \rho_0^2).$$

Hence, from (7), we have

$$\|u(\mu) - u_0\|_X \leq C \|\partial_X J(u_0, \mu)\|_{X'} \leq 2C\omega(\rho)\rho = o(\rho).$$

□

Under the conditions of Theorem 2.4,  $\partial_X^2 J(u(\mu), \mu)$  can be regarded as a linear topological isomorphism from  $X$  to  $X'$  from the Lax-Milgram theorem. Therefore, we can define  $\Lambda(\mu) \in B(X', X)$  which satisfies

$$\partial_X^2 J(u(\mu), \mu)[\Lambda(\mu)h, w] = h[w] \quad (\forall w \in X, \forall h \in X').$$

The Fréchet derivative of the local minimizer  $u(\mu)$  with respect to parameter  $\mu$  is given by the next proposition.

**Proposition 2.8.** *Under the condition of Theorem 2.5 with  $k = 1$ ,*

$$(8) \quad u'(\mu) = -\Lambda(\mu)h_0(\mu) \quad (\mu \in \mathcal{O}_0),$$

*holds, where  $h_0(\mu) := \partial_M \partial_X J(u(\mu), \mu) \in B(M, X')$ .*

*Proof.* Differentiating  $\partial_X J(u(\mu), \mu) = 0 \in X'$  by  $\mu$ , we have

$$\partial_X^2 J(u(\mu), \mu)[u'(\mu)] + \partial_M \partial_X J(u(\mu), \mu) = 0 \in B(M, X').$$

This is equivalent to (8) from the Lax-Milgram theorem. □

**Proposition 2.9.** *Under the condition of Theorem 2.4, we additionally assume that  $J \in C^2(\mathcal{U} \times \mathcal{O})$  then  $J_* \in C^2(\mathcal{O}_0)$  and it satisfies*

$$\begin{aligned} J_*''(\mu)[\mu_1, \mu_2] = & \partial_M^2 J(u(\mu), \mu)[\mu_1, \mu_2] \\ & - {}_X \langle \Lambda(\mu)h_0(\mu)[\mu_1], h_0(\mu)[\mu_2] \rangle_{X'}, \end{aligned}$$

*for  $\mu \in \mathcal{O}_0$  and  $\mu_1, \mu_2 \in M$ .*

*Proof.* Differentiating the formula (5) by  $\mu$  and substituting (8), we obtain the formula. □

### 3. LIPSCHITZ DEFORMATION OF DOMAINS

We study a domain deformation with Lipschitz transform  $\varphi : \Omega \rightarrow \varphi(\Omega)$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  ( $n \in \mathbb{N}$ ) and  $\varphi$  is a  $\mathbb{R}^n$ -valued Lipschitz function. The identity map on  $\mathbb{R}^n$  is denoted by  $\varphi_0(x) = x$  ( $x \in \mathbb{R}^n$ ).

For a function  $u : \Omega \rightarrow \mathbb{R}^k$ , we define

$$|u|_{\text{Lip}, \Omega} := \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|},$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^n$  or  $\mathbb{R}^k$ . If  $|u|_{\text{Lip}, \Omega} < \infty$ ,  $u$  is called uniformly Lipschitz continuous on  $\Omega$ . It is known that, for  $u \in W^{1, \infty}(\Omega)$ , there is  $\tilde{u} \in C^0(\Omega)$  such that  $\tilde{u}(x) = u(x)$  a.e.  $x \in \Omega$ , in other words, we can regard  $W^{1, \infty}(\Omega) \subset C^0(\Omega)$ . If  $\Omega$  is convex,  $W^{1, \infty}(\Omega) = C^{0,1}(\bar{\Omega})$  as a subset of  $C^0(\Omega)$ . Moreover, if  $k = 1$ , we have

$$\|\nabla u\|_{L^\infty(\Omega)} = |u|_{\text{Lip}, \Omega} \quad (u \in W^{1, \infty}(\Omega) \cap C^0(\Omega)).$$

In the following argument, we fix a bounded convex domain  $\Omega_0 \subset \mathbb{R}^n$  ( $n \geq 2$ ), and we identify  $W^{1, \infty}(\Omega_0, \mathbb{R}^n)$  with  $C^{0,1}(\bar{\Omega}_0, \mathbb{R}^n)$ .

**Proposition 3.1.** *Suppose that  $\varphi \in W^{1, \infty}(\Omega_0, \mathbb{R}^n)$  satisfies  $|\varphi - \varphi_0|_{\text{Lip}, \Omega_0} < 1$ . Then  $\varphi$  is a bi-Lipschitz transform from  $\Omega_0$  to  $\varphi(\Omega_0)$ , i.e.  $\varphi$  is bijective from  $\Omega_0$  onto an open set and  $\varphi$  and  $\varphi^{-1}$  are both uniformly Lipschitz continuous. Moreover, we have*

$$(9) \quad \text{ess-inf}_{\Omega_0} (\det \nabla \varphi^T) \geq (1 - |\varphi - \varphi_0|_{\text{Lip}, \Omega_0})^n > 0.$$

*where  $\nabla \varphi^T$  is the Jacobian matrix defined by*

$$\nabla \varphi^T(x) := \left( \frac{\partial \varphi_j}{\partial x_i}(x) \right)_{i \downarrow, j \rightarrow} \in \mathbb{R}^{n \times n},$$

*for  $x = (x_1, \dots, x_n)^T \in \Omega_0$ .*

*Proof.* Let  $\mu := \varphi - \varphi_0$  and  $\theta := |\mu|_{\text{Lip}, \Omega_0} \in (0, 1)$ . First, we show that  $\varphi(\Omega_0)$  is open. We arbitrarily fix  $x_0 \in \Omega_0$  and define  $y_0 := \varphi(x_0)$ . Let  $\delta > 0$  such that  $\overline{B_\delta(x_0)} \subset \Omega_0$ , where  $B_\delta(x_0) := \{x \in \mathbb{R}^n; |x - x_0| < \delta\}$ . For  $y \in B_{(1-\theta)\delta}(y_0)$ , we show that  $y \in \varphi(\Omega_0)$ . It is easily checked that  $T(\xi) := y - \mu(\xi)$  is a uniform contraction on  $\overline{B_\delta(x_0)}$ . From the contraction mapping theorem, there is a fixed point  $x = T(x) = y - \mu(x)$  in  $\overline{B_\delta(x_0)}$ , that is  $y = \varphi(x)$ . Hence,  $\varphi(\Omega_0)$  is an open set. Since  $|\varphi(x_1) - \varphi(x_2)| \geq |x_1 - x_2| - |\mu(x_1) - \mu(x_2)| \geq (1-\theta)|x_1 - x_2|$  for  $x_1, x_2 \in \Omega_0$ , it follows that  $\varphi$  is injective and  $\varphi^{-1}$  satisfies uniform Lipschitz condition on  $\varphi(\Omega_0)$ .

From Rademacher's theorem (see [5], [23]),  $\mu$  is differentiable almost everywhere and the derivative coincides with the distributional derivative almost everywhere, i.e., there exists  $\mathcal{N} \subset \Omega_0$  with  $\mathcal{L}^n(\mathcal{N}) = 0$  such that

$$\nabla^T \mu(x)y = \lim_{h \rightarrow 0} \frac{\mu(x+hy) - \mu(x)}{h} \quad (x \in \Omega_0 \setminus \mathcal{N}),$$

where  $\nabla^T \mu = (\nabla \mu^T)^T$ . It follows that

$$|\nabla^T \mu(x)y| \leq \theta |y| \quad (x \in \Omega_0 \setminus \mathcal{N}),$$

and that the moduli of all the eigenvalues of  $\nabla^T \mu(x)$  for  $x \in \Omega_0 \setminus \mathcal{N}$  are bounded by  $\theta$ . Hence, we obtain

$$\det(\nabla^T \varphi(x)) = \det(I + \nabla^T \mu(x)) \geq (1-\theta)^n \quad (x \in \Omega_0 \setminus \mathcal{N}).$$

□

We fix an open set  $\Omega$  which satisfies  $\overline{\Omega} \subset \Omega_0$  and the deformed domain  $\varphi(\Omega)$  is denoted by  $\Omega(\varphi)$  under the condition of Proposition 3.1, hereafter. We define a pushforward operator  $\varphi_*$  which transforms a function  $v$  on  $\Omega$  to a function  $\varphi_* v := v \circ \varphi^{-1}$  on  $\Omega(\varphi)$ , if  $\varphi$  satisfies Proposition 3.1. We define

$$\begin{aligned} A(\varphi) &:= (\nabla \varphi^T)^{-1} \in L^\infty(\Omega_0, \mathbb{R}^{n \times n}), \\ \kappa(\varphi) &:= \det \nabla \varphi^T \in L^\infty(\Omega_0, \mathbb{R}). \end{aligned}$$

These Jacobi matrices and Jacobian appear in the pull-back of differentiation and integration on  $\Omega(\varphi)$  to  $\Omega$ . For a function  $v$  on  $\Omega$ , we have

$$(10) \quad [\nabla(\varphi_* v)] \circ \varphi = A(\varphi) \nabla v \quad \text{a.e. in } \Omega \quad (v \in W^{1,1}(\Omega)),$$

$$(11) \quad \int_{\Omega(\varphi)} (\varphi_* v)(y) dy = \int_{\Omega} v(x) \kappa(\varphi)(x) dx \quad (v \in L^1(\Omega)).$$

These equalities are well known in the case  $\varphi \in C^1$ . However, for  $\varphi \in C^{0,1}$ , these are not so trivial. See, [5] and [23] etc. for details. We omit the proof of the next proposition since it is clear from (10) and (11).

**Proposition 3.2.** *Under the condition of Proposition 3.1, for  $p \in [1, \infty]$ ,  $\varphi_*$  is a linear topological isomorphism from  $L^p(\Omega)$  onto  $L^p(\Omega(\varphi))$ , and a linear topological isomorphism from  $W^{1,p}(\Omega)$  onto  $W^{1,p}(\Omega(\varphi))$ .*

The following theorem plays an essential role in the application to the shape derivatives.

**Theorem 3.3.** *Let  $\Omega$  be an open subset of  $\Omega_0$ .*

1.  $\kappa \in C^\infty(W^{1,\infty}(\Omega, \mathbb{R}^n), L^\infty(\Omega))$ , and  $\kappa'(\varphi_0)[\mu] = \text{div} \mu$  for  $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ .
2.  $A \in C^\infty(\mathcal{O}, L^\infty(\Omega, \mathbb{R}^{n \times n}))$ , where

$$\mathcal{O} := \{\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^n); \text{ess-inf}_\Omega \kappa(\varphi) > 0\}.$$

*In particular, it holds that  $A'(\varphi_0)[\mu] = -\nabla \mu^T$  for  $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ .*

*Proof.* Since the determinant is a polynomial of degree  $n$ , it is clear that  $\kappa$  belongs to  $C^\infty(W^{1,\infty}(\Omega, \mathbb{R}^n), L^\infty(\Omega))$ . For fixed  $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ , we define

$$m_{ij}(t) := \delta_{ij} + t \frac{\partial \mu_j}{\partial x_i} \in L^\infty(\Omega) \quad (i, j = 1, \dots, n, t \in \mathbb{R}),$$

where  $\delta_{ij}$  is the Kronecker's delta. Then we have

$$\begin{aligned} \kappa'(\varphi_0)[\mu] &= \frac{d}{dt} \Big|_{t=0} \kappa(\varphi_0 + t\mu) = \frac{d}{dt} \Big|_{t=0} \det(m_{ij}(t)) \\ &= \frac{d}{dt} \Big|_{t=0} \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) m_{1\sigma(1)}(t) \cdots m_{n\sigma(n)}(t) \right) \\ &= \frac{d}{dt} \Big|_{t=0} (m_{11}(t) \cdots m_{nn}(t)) \\ &= \sum_{i=1}^n m_{11}(0) \cdots m'_{ii}(0) \cdots m_{nn}(0) = \text{div} \mu. \end{aligned}$$

Let the  $(i, j)$  component of  $A(\varphi)$  be denoted by  $a_{ij}(\varphi) \in L^\infty(\Omega)$ . Then we have  $a_{ij}(\varphi) = \alpha_{ij}(\varphi)/\kappa(\varphi)$ , where  $\alpha_{ij}(\varphi)$  is the  $(i, j)$  cofactor of  $\nabla \varphi^T$ , which is a polynomial of  $\frac{\partial \varphi_k}{\partial x_l}$  of degree  $n-1$ . Since  $\text{ess-inf}_\Omega \kappa(\varphi) > 0$  for  $\varphi \in \mathcal{O}$ ,  $a_{ij} \in C^\infty(\mathcal{O}, L^\infty(\Omega))$  follows. For fixed  $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ , differentiating the identity

$$A(\varphi_0 + t\mu)(I + t\nabla \mu^T) = I \quad (I: \text{identity matrix of degree } n),$$

by  $t \in \mathbb{R}$  at  $t = 0$ , we have

$$A'(\varphi_0)[\mu] + A(\varphi_0) \nabla \mu^T = O.$$

Since  $A(\varphi_0) = I$ , we have  $A'(\varphi_0)[\mu] = -\nabla \mu^T$ . □

We define an open subset of  $W^{1,\infty}(\Omega_0, \mathbb{R}^n)$  as

$$(12) \quad \mathcal{O}(\Omega) := \left\{ \varphi \in W^{1,\infty}(\Omega_0, \mathbb{R}^n); \begin{array}{l} \overline{\Omega(\varphi)} \subset \Omega_0 \\ |\varphi - \varphi_0|_{\text{Lip}, \Omega_0} < 1 \end{array} \right\}.$$

**Proposition 3.4.** *We assume  $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$  with  $\text{supp}(\mu) \subset \Omega$ .*

1. *If  $|\mu|_{\text{Lip}, \Omega_0} < 1$ , then  $\varphi = \varphi_0 + \mu$  is a bi-Lipschitz transform from  $\Omega$  onto itself.*
2. *For  $t \in \mathbb{R}$  with  $|t\mu|_{\text{Lip}, \Omega_0} < 1$ , we define a bi-Lipschitz transform  $\varphi(t) = \varphi_0 + t\mu$  from  $\Omega$  to itself. Let  $l \in \{0, 1\}$  and  $p \in [1, \infty]$ . Suppose that  $f \in W^{l,p}(\Omega)$  if  $p \in [1, \infty)$ , and  $f \in C^l(\Omega) \cap W^{l,\infty}(\Omega)$  if  $p = \infty$ . Then  $\varphi(t)_* f \rightarrow f$  strongly in  $W^{l,p}(\Omega)$  as  $t \rightarrow 0$ .*

*Proof.* From Proposition 3.1, claim 1 is clear. For claim 2, let us fix  $t_0 > 0$  with  $|t_0\mu|_{Lip, \Omega_0} < 1$ . Then, from Proposition 3.2, there exist  $C > 0$  such that the following inequalities hold for  $|t| \leq t_0$ ,

$$\begin{aligned} \|\varphi(t)_*f - f\|_{W^{l,p}(\Omega)} &= \|\varphi(t)_*(f - f \circ \varphi(t))\|_{W^{l,p}(\Omega)} \\ &\leq C\|f - f \circ \varphi(t)\|_{W^{l,p}(\Omega)}. \end{aligned}$$

Since  $[\varphi \mapsto f \circ \varphi] \in C^0(\mathcal{O}(\Omega), W^{l,p}(\Omega))$ , we obtain

$$\|f - f \circ \varphi(t)\|_{W^{l,p}(\Omega)} = \|f \circ \varphi_0 - f \circ \varphi(t)\|_{W^{l,p}(\Omega)} \rightarrow 0,$$

as  $t \rightarrow 0$ . □

#### 4. VARIATION OF ELASTIC ENERGY

We consider an application of our abstract parameter variation formula stated in Section 2 to shape variation of elastic energy with an inhomogeneous and anisotropic elastic tensor.

Let  $\Omega_0$  be a fixed bounded convex open set of  $\mathbb{R}^n$  ( $n \geq 2$ ). We consider an elastic body denoted by  $\Omega$ , which is an open subset of  $\Omega_0$  with  $\bar{\Omega} \subset \Omega_0$ . For a small displacement vector field  $u = (u_1, \dots, u_n)^T \in H^1(\Omega)^n$ , the strain tensor  $e[u](x) = (e_{ij}[u](x)) \in \mathbb{R}_{sym}^{n \times n}$  and the stress tensor  $\sigma[u](x) = (\sigma_{ij}[u](x)) \in \mathbb{R}_{sym}^{n \times n}$  are given by the strain-displacement relations and the constitutive equations (generalized Hooke's law):

$$(13) \quad e[u] := \frac{1}{2}(\nabla^T u + \nabla u^T), \quad \text{i.e.} \quad e_{ij}[u] = \frac{1}{2}(\partial_j u_i + \partial_i u_j).$$

$$\sigma[u](x) = C(x)e[u](x) \quad \text{i.e.} \quad \sigma_{ij}(x) = c_{ijkl}(x)e_{kl}(x),$$

where  $C(x) = (c_{ijkl}(x)) \in \mathbb{R}^{n \times n \times n \times n}$  is the (anisotropic) elasticity tensor with the symmetries  $c_{ijkl} = c_{klij} = c_{jikl}$  ( $i, j, k, l = 1, \dots, n$ ). It should satisfy the positivity condition:

$$(14) \quad \exists c_* > 0 \quad \text{s.t.} \quad c_{ijkl}(x) \xi_{ij} \xi_{kl} \geq c_* |\xi|^2 \quad (\forall x \in \Omega_0, \forall \xi \in \mathbb{R}_{sym}^{n \times n}).$$

where  $|\xi| := \sqrt{\xi_{ij} \xi_{ij}}$ . It depends on the elastic property of the material  $\Omega$ , and is supposed to be given. If the material is homogeneous, the elasticity tensor should be constant  $C(x) \equiv C$ . From the strain-displacement relation (13), we write  $\sigma[u] := C e[u]$ .

The displacement field  $u(x)$  should satisfy the the equilibrium equations of force and some boundary conditions. They consist of the following linear second order elliptic boundary value problem.

$$(15) \quad \begin{cases} -\operatorname{div} \sigma[u] = f(x) & (x \in \Omega) \\ u = g(x) & (x \in \Gamma_D) \\ -\sigma[u] \nu = h(x) & (x \in \Gamma_N). \end{cases}$$

In the first equation,  $\operatorname{div} \sigma$  means  $\operatorname{div} \sigma = (\partial_i \sigma_{i1}, \dots, \partial_i \sigma_{in})^T$  and  $f(x) = (f_1(x), \dots, f_n(x))^T \in \mathbb{R}^n$  stands for the given body force.

The boundary  $\Gamma := \partial\Omega$  is divided into two parts as

$$\Gamma = \Gamma_D \cup \Gamma_N, \quad \Gamma_D \cap \Gamma_N = \emptyset, \quad |\Gamma_D| > 0,$$

(on the other hand,  $\Gamma_N$  can be empty). The outward unit normal vector to  $\Gamma$  at  $x \in \Gamma$  is denoted by  $\nu(x) \in \mathbb{R}^n$ . For the boundary condition of displacement on  $\Gamma_D$ , we suppose that  $g \in H^1(\Omega, \mathbb{R}^n)$  exists and the boundary condition is given by its boundary value  $g|_{\Gamma_D}$  in the sense of the trace operator. The surface force on  $\Gamma_N$ , is given by  $h \in L^2(\Gamma_N, \mathbb{R}^n)$ , as a prescribed boundary value.

The strain energy density  $w[u](x) \in \mathbb{R}$  is defined by

$$\begin{aligned} w[u](x) &= \frac{1}{2} \sigma[u](x) : e[u](x) := \frac{1}{2} \operatorname{tr}(\sigma[u](x)e[u](x)) \\ \text{i.e.} \quad w[u](x) &= \frac{1}{2} \sigma_{ij}[u](x)e_{ij}[u](x). \end{aligned}$$

If  $e[u](x) \neq 0$ ,  $w[u](x) > 0$  follows from the the positivity condition of  $C(x)$ .

For the problem (15), we introduce the total elastic energy:

$$E(v) := \int_{\Omega} \{w[v] - f \cdot v\} dx + \int_{\Gamma_N} h \cdot v d\mathcal{H}^{n-1} \quad (v \in H^1(\Omega, \mathbb{R}^n)).$$

For a fixed  $\Omega$  and  $\Gamma_D$ , we define the affine space

$$V(g) := \{v \in H^1(\Omega, \mathbb{R}^n); v|_{\Gamma_D} = g|_{\Gamma_D}\},$$

for  $g \in H^1(\Omega, \mathbb{R}^n)$ , and we set a closed subspace  $V := V(0)$ . On the geometry of  $\Omega$  and  $\Gamma_D$ , we assume the following condition:

$$(16) \quad \exists c_0 > 0, \text{ s.t. } \int_{\Omega} |e[v]|^2 dx \geq c_0 \|v\|_{H^1(\Omega, \mathbb{R}^n)}^2 \quad (v \in V).$$

The condition (16) is called Korn's inequality and is satisfied by most domains under the positivity condition (14). See details for Korn's inequality in [11] and [13] etc.

We consider the following minimization problem:

$$\text{minimize } E(v) \text{ among } v \in V(g)$$

Owing to the Korn's inequality (16), the functional  $E(v)$  is strictly convex and admits a unique minimizer  $\bar{u}_0 \in V(g)$ . The first variation of this minimizing problem gives us the following equality.

$$(17) \quad \int_{\Omega} \sigma[\bar{u}_0] : e[v] dx - \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} h \cdot v d\mathcal{H}^{n-1} = 0 \quad (\forall v \in V).$$

For  $u \in V(g)$ ,  $u$  is the unique minimizer of  $E(\cdot)$  in  $V(g)$  if and only if  $u = \bar{u}_0$  satisfies (17). Moreover, (17) means that  $\bar{u}_0$  is a weak solution of (15).

We consider a family of minimization problem parametrized by  $\varphi \in \mathcal{O}'(\Omega)$ :

$$\mathcal{O}'(\Omega) := \{\varphi \in \mathcal{O}(\Omega); \varphi(x) = x, (x \in \operatorname{supp}(h) \cap \Gamma_N)\},$$

where  $\mathcal{O}(\Omega)$  is defined by (12). For  $\varphi \in \mathcal{O}'(\Omega)$ , we define a total elastic energy and an affine space in the modified domain  $\Omega(\varphi)$ :

$$E(v, \varphi) := \int_{\Omega(\varphi)} \{w[v] - f \cdot v\} dx + \int_{\Gamma_N} h \cdot v d\mathcal{H}^{n-1} \\ (v \in H^1(\Omega(\varphi), \mathbb{R}^n)).$$

$$V(g, \varphi) := \varphi_*(V(g)) \\ = \{v \in H^1(\Omega(\varphi), \mathbb{R}^n); \varphi_*^{-1}(v) - g \in V\} \\ (g \in H^1(\Omega, \mathbb{R}^n)).$$

**Problem 4.1.** For given  $\varphi \in \mathcal{O}'(\Omega)$  and  $g \in H^1(\Omega, \mathbb{R}^n)$ , find a global minimizer  $\bar{u}(\varphi)$  of  $E(\cdot, \varphi)$  in  $V(g, \varphi)$

We define an elastic potential energy in the modified domain  $\Omega(\varphi)$ :

$$(18) \quad E_*(\varphi) := E(\bar{u}(\varphi), \varphi),$$

and consider its Fréchet differentiability with respect to the parameter  $\varphi$  in this section.

We denote the pullbacks of the strain and stress tensors, and the strain energy density of  $\varphi_*v$  for  $v \in H^1(\Omega, \mathbb{R}^n)$  by

$$e[v, \varphi] := \varphi_*^{-1}(e[\varphi_*v]), \quad \sigma[v, \varphi] := \varphi_*^{-1}(\sigma[\varphi_*v]), \\ w[v, \varphi] := \varphi_*^{-1}(w[\varphi_*v]).$$

Then we have

$$e[v, \varphi] = \frac{1}{2} \{(A(\varphi)\nabla v^T)^T + A(\varphi)\nabla v^T\} \\ \sigma[v, \varphi] = (C \circ \varphi)e[v, \varphi] \\ w[v, \varphi] = \frac{1}{2} \sigma[v, \varphi] : e[v, \varphi]$$

We define a modified elastic energy in the original domain:

$$\mathcal{E}_0(v, \varphi) := \int_{\Omega(\varphi)} \{w[\varphi_*v] - f \cdot \varphi_*v\} dx \\ (v \in H^1(\Omega, \mathbb{R}^n), \varphi \in \mathcal{O}(\Omega)).$$

For a simple notation, we also define

$$W(\xi, \eta, \zeta) := \frac{1}{2} c_{ijkl}(\xi) \zeta_{ij} \zeta_{kl} - f_i(\xi) \eta_i \\ (\xi \in \Omega_0, \eta \in \mathbb{R}^n, \zeta \in \mathbb{R}_{\text{sym}}^{n \times n}).$$

Then, the modified elastic energy is also written in the form:

$$\mathcal{E}_0(v, \varphi) = \int_{\Omega(\varphi)} W(x, \varphi_*v, e[\varphi_*v]) dx \\ = \int_{\Omega} W(\varphi(x), v, e[v, \varphi]) \kappa(\varphi) dx$$

The derivatives of  $W$  are denoted as follows:

$$\nabla_{\xi} W(\xi, \eta, \zeta) = \frac{1}{2} \nabla c_{ijkl}(\xi) \zeta_{ij} \zeta_{kl} - \nabla f^T(\xi) \eta, \\ \nabla_{\eta} W(\xi, \eta, \zeta) = -f(\xi), \\ \frac{\partial W}{\partial \zeta_{ij}}(\xi, \eta, \zeta) := \frac{d}{dt} W(\xi, \eta, \zeta + t\zeta^{(ij)}) \Big|_{t=0} = c_{ijkl}(\xi) \zeta_{kl},$$

where  $\zeta^{(ij)} = (\zeta_{kl}^{(ij)}) \in \mathbb{R}^{n \times n}$  is defined by  $\zeta_{kl}^{(ij)} = \delta_{ik} \delta_{jl}$ . For  $v \in H^1(\Omega, \mathbb{R}^n)$ , we write  $W(v) = W(x, v(x), e[v](x))$ ,  $\nabla_{\xi} W(v) = \nabla_{\xi} W(x, v(x), e[v](x))$ , etc. We remark that

$$\frac{\partial W}{\partial \zeta_{ij}}(v) = \frac{\partial W}{\partial \zeta_{ij}}(x, v(x), e[v](x)) = \sigma_{ij}[v].$$

**Lemma 4.2.** For  $v \in H^1(\Omega, \mathbb{R}^n)$  and  $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ ,

$$\frac{d}{dt} W(x, v(x), e[v, \varphi_0 + t\mu]) \Big|_{t=0} \\ = -\sigma[v](x) : (\nabla \mu^T(x) \nabla v^T(x)) \quad (\text{a.e. } x \in \Omega)$$

holds.

*Proof.* From Theorem 3.3, we have

$$\frac{d}{dt} e[v, \varphi_0 + t\mu](x) \Big|_{t=0} \\ = \frac{1}{2} \frac{d}{dt} \{(A(\varphi_0 + t\mu)\nabla v^T)^T + A(\varphi_0 + t\mu)\nabla v^T\} \Big|_{t=0} \\ = -\frac{1}{2} \{(\nabla \mu^T \nabla v^T)^T + \nabla \mu^T \nabla v^T\}.$$

Hence we obtain

$$\frac{d}{dt} W(x, v(x), e[v, \varphi_0 + t\mu]) \Big|_{t=0} \\ = \frac{\partial W}{\partial \zeta_{ij}}(v) \frac{d}{dt} e_{ij}[v, \varphi_0 + t\mu] \Big|_{t=0} \\ = -\frac{1}{2} \sigma[v] : \{(\nabla \mu^T \nabla v^T)^T + \nabla \mu^T \nabla v^T\}. \\ = -\sigma[v] : (\nabla \mu^T(x) \nabla v^T(x)),$$

for a.e.  $x \in \Omega$  □

**Lemma 4.3.** Let  $m \in \mathbb{N} \cup \{0\}$ , and let  $q > 1$ . We assume that  $q \geq \frac{2n}{n+2}$  if  $\Omega$  satisfies the cone property. If the cone property does not hold for  $\Omega$ , we set  $q = 2$ . Then, under the condition:

$$(19) \quad c_{ijkl} \in C^m(\Omega_0), \quad f \in W^{m,q}(\Omega_0, \mathbb{R}^n),$$

the following regularities hold:

$$\mathcal{E}_0(\cdot, \varphi) \in C^\infty(H^1(\Omega, \mathbb{R}^n)) \quad (\varphi \in \mathcal{O}(\Omega)), \\ \mathcal{E}_0 \in C^m(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega)), \\ \partial_v \mathcal{E}_0 \in C^m(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega), H^1(\Omega, \mathbb{R}^n)'), \\ \partial_v^2 \mathcal{E}_0 \text{ is continuous on } H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega)$$

*Proof.* Using the formulas (10) and (11), we have

$$\mathcal{E}_0(v, \varphi) = \int_{\Omega(\varphi)} \varphi_* \{w[v, \varphi] - (f \circ \varphi) \cdot v\} dx \\ = \int_{\Omega} \{w[v, \varphi] - (f \circ \varphi) \cdot v\} \kappa(\varphi) dx.$$

Under the assumptions, from Theorem 3.3, we obtain

$$[(v, \varphi) \mapsto A(\varphi)\nabla v^T] \in C^\infty(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega), L^2(\Omega, \mathbb{R}^{n \times n})), \\ [(v, \varphi) \mapsto e[v, \varphi]] \in C^\infty(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega), L^2(\Omega, \mathbb{R}^{n \times n})),$$

$$[\varphi \mapsto C \circ \varphi] \in C^m(\mathcal{O}(\Omega), L^\infty(\Omega, \mathbb{R}^{n \times n \times n \times n})),$$

$$[(v, \varphi) \mapsto \sigma[v, \varphi]] \in C^m(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega), L^2(\Omega, \mathbb{R}^{n \times n})).$$

From these regularities, it follows that

$$[(v, \varphi) \mapsto w[v, \varphi]] \in C^k(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega), L^1(\Omega)).$$

We define  $q^* > 1$  by the condition  $1/q + 1/q^* = 1$ . From the Sobolev embedding theorem (see [1] etc.),  $H^1(\Omega)$  is continuously embedded in  $L^{q^*}(\Omega)$ . From the regularities:

$$[\varphi \mapsto f \circ \varphi] \in C^m(\mathcal{O}(\Omega), L^q(\Omega, \mathbb{R}^n)),$$

$$[(f, v) \mapsto f \cdot v] \in C^\infty(L^q(\Omega, \mathbb{R}^n) \times H^1(\Omega, \mathbb{R}^n), L^1(\Omega)),$$

and  $\kappa \in C^\infty(\mathcal{O}(\Omega), L^\infty(\Omega))$  (Theorem 3.3), we conclude that  $\mathcal{E}_0 \in C^m(H^1(\Omega, \mathbb{R}^n) \times \mathcal{O}(\Omega))$ .

Since  $\mathcal{E}_0(v, \varphi)$  is quadratic with respect to  $v$ ,  $\mathcal{E}_0(\cdot, \varphi) \in C^\infty(H^1(\Omega, \mathbb{R}^n))$  is clear. The Fréchet derivatives of  $\mathcal{E}_0$  are

$$\begin{aligned} \partial_v \mathcal{E}_0(v, \varphi)[w] &= \int_{\Omega(\varphi)} \{\sigma[\varphi_* v] : e[\varphi_* w] - f \cdot (\varphi_* w)\} dx \\ &= \int_{\Omega} \{\sigma[v, \varphi] : e[w, \varphi] - (f \circ \varphi) \cdot w\} \kappa(\varphi) dx, \end{aligned}$$

$$\partial_v^2 \mathcal{E}_0(v, \varphi)[w_1, w_2] = \int_{\Omega} \{\sigma[w_1, \varphi] : e[w_2, \varphi]\} \kappa(\varphi) dx.$$

From these expressions, the regularities for  $\partial_v \mathcal{E}_0$  and  $\partial_v^2 \mathcal{E}_0$  follow similarly.  $\square$

For  $v \in H^1(\Omega, \mathbb{R}^n)$  and  $\mu \in \mathcal{O}(\Omega)$ , we define

$$Q[v, \mu](x) := \nabla_\xi W(v) \cdot \mu - \sigma[v] : (\nabla \mu^T \nabla v^T) + W(v) \operatorname{div} \mu.$$

Then we have the following lemma.

**Lemma 4.4.** *Under the condition of Lemma 4.3, if  $m \geq 1$ , the following formula holds:*

$$\partial_\varphi \mathcal{E}_0(v, \varphi_0)[\mu] = \int_{\Omega} Q[v, \mu] dx \quad (v \in H^1(\Omega, \mathbb{R}^n), \mu \in \mathcal{O}(\Omega)).$$

*Proof.* Since  $\mathcal{E}$  is Fréchet differentiable, it holds that

$$\begin{aligned} \partial_\varphi \mathcal{E}_0(v, \varphi_0)[\mu] &= \frac{d}{dt} \mathcal{E}_0(v, \varphi_0 + t\mu) \Big|_{t=0} \\ &= \frac{d}{dt} \int_{\Omega} W(x + t\mu(x), v, e[v, \varphi_0 + t\mu]) \kappa(\varphi_0 + t\mu) dx \Big|_{t=0} \\ &= \int_{\Omega} \{\nabla_\xi W(v) \cdot \mu - \sigma[v] : (\nabla \mu^T \nabla v^T)\} \kappa(\varphi_0) dx \\ &\quad + \int_{\Omega} W(v) \kappa'(\varphi_0)[\mu] dx \\ &= \int_{\Omega} \{\nabla_\xi W(v) \cdot \mu - \sigma[v] : (\nabla \mu^T \nabla v^T) + W(v) \operatorname{div} \mu\} dx. \end{aligned}$$

$\square$

We remark that in case of a translation of domain, i.e. if  $\mu$  is a constant vector,  $Q[v, \mu]$  becomes

$$Q[v, \mu](x) = \nabla_\xi W(v) \cdot \mu.$$

Moreover, if the material is homogeneous, i.e. if  $c_{ijkl}$  and  $f_j$  are constant, then  $\nabla_\xi W(v) = 0$  in  $\Omega$  and

$$\partial_\varphi \mathcal{E}(v, \varphi_0)[\mu] = 0$$

holds. Since a homogeneous media is invariant under a translation, this is a natural result.

We define

$$\begin{aligned} \mathcal{E}(v, \varphi) &:= E(\varphi_*(v + g), \varphi) \\ (20) \quad &= \mathcal{E}_0(v + g, \varphi) + \int_{\Gamma_N} h \cdot (v + g) d\mathcal{H}^{n-1} \\ &\quad (v \in V, \varphi \in \mathcal{O}(\Omega)). \end{aligned}$$

Applying the abstract theory in Section 2 to  $\mathcal{E}$ , we have the following theorem.

**Theorem 4.5.** *Suppose the conditions of Lemma 4.3 and (16). Then there exists an open subset  $\mathcal{O}_0 \subset \mathcal{O}(\Omega)$  with  $\varphi_0 \in \mathcal{O}_0$  such that, for  $\varphi \in \mathcal{O}_0$ , there exists a unique minimizer  $\bar{u}(\varphi) \in V(g, \varphi)$  of  $E(\cdot, \varphi)$ . In addition, the mapping  $[\varphi \mapsto \varphi_*^{-1}(\bar{u}(\varphi))]$  belongs to  $C^m(\mathcal{O}_0, H^1(\Omega, \mathbb{R}^n))$  and  $E_* \in C^m(\mathcal{O}_0)$ . Furthermore, if  $m \geq 1$ , the formula:*

$$E'_*(\varphi_0)[\mu] = \int_{\Omega} Q[\bar{u}_0, \mu](x) dx \quad (\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n))$$

holds.

*Proof.* We apply Theorems 2.4-2.6 to the functional  $J = \mathcal{E}$  with  $\mathcal{U} = X = V$ ,  $M = W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ ,  $\mathcal{O} = \mathcal{O}(\Omega)$ ,  $u_0 = \bar{u}_0 - g$ , and  $\mu_0 = \varphi_0$ . From Lemma 4.3 and (20), the regularity conditions in Theorems 2.4-2.6 are all satisfied. The condition 4 in Theorem 2.4 is nothing but the Korn's inequality (16).

From Theorem 2.4 and Theorem 2.5, there exists  $\mathcal{O}_0 \subset \mathcal{O}(\Omega)$  and  $u \in C^m(\mathcal{O}_0, V)$  such that

$$\bar{u}(\varphi) := \varphi_*(u(\varphi) + g) \in V(g, \varphi),$$

gives a unique minimizer of  $E(\cdot, \varphi)$  for  $\varphi \in \mathcal{O}_0$ . Since  $\mathcal{E}(v, \varphi)$  is a quadratic polynomial with respect to  $v \in V$ , it is clear that  $\mathcal{U}_0$  can be chosen as  $\mathcal{U}_0 = V$ .

We define

$$\mathcal{E}_*(\varphi) := \mathcal{E}(u(\varphi), \varphi) \quad (\varphi \in \mathcal{O}_0).$$

Then it follows that

$$E_*(\varphi) = E(\bar{u}(\varphi), \varphi) = \mathcal{E}_*(\varphi).$$

From Theorem 2.6, we obtain  $E_* = \mathcal{E}_* \in C^m(\mathcal{O}_0)$ . Furthermore, if  $m \geq 1$ ,

$$\begin{aligned} E'_*(\varphi_0) &= \mathcal{E}'_*(\varphi_0) = \partial_\varphi \mathcal{E}(u(\varphi_0), \varphi_0) \\ &= \partial_\varphi \mathcal{E}_0(u(\varphi_0) + g, \varphi_0) = \partial_\varphi \mathcal{E}_0(\bar{u}_0, \varphi_0), \end{aligned}$$



holds. Applying Lemma 4.4, we obtain the formula for  $E'_*(\varphi_0)$ .  $\square$

The weak solution for (15), which was given as a minimizer  $\bar{u}_0 = u(\varphi_0) + g$ , is denoted simply by  $u$  hereafter. We also set  $m = 1$  and  $q = 2$  in the condition (19) hereafter just for simplicity, i.e., we suppose

$$(21) \quad c_{ijkl} \in C^1(\Omega_0), \quad f \in H^1(\Omega_0, \mathbb{R}^n).$$

The following theorem gives a formula of integration by parts for  $Q[v, \mu]$ .

**Theorem 4.6.** *Suppose the condition (21). Let  $D$  be a subdomain of  $\Omega$  with the cone property, in which the Gauss-Green formula holds. Then, for  $v \in H^2(D, \mathbb{R}^n)$  and  $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ , the following formula holds:*

$$\begin{aligned} \int_D Q[v, \mu] dx &= \int_D (\operatorname{div} \sigma[v] + f) \cdot ((\mu \cdot \nabla)v) dx \\ &+ \int_{\partial D} \{W(v)\mu \cdot \nu - (\sigma[v]\nu) \cdot ((\mu \cdot \nabla)v)\} d\mathcal{H}^{n-1}, \end{aligned}$$

where  $\nu$  stands for the unit outward normal vector on  $\partial D$ .

*Proof.* Since  $e_{ij}[v] \in H^1(D)$ , from the Sobolev embedding theorem,  $W(v) \in W^{1,s}(D)$  holds for some  $s > 1$ . Hence, we can apply the Gauss-Green formula and obtain

$$(22) \quad \begin{aligned} &\int_D Q[v, \mu] dx \\ &= \int_D \{(\nabla_\xi W(v) - \nabla_x[W(v)]) \cdot \mu - \sigma[v] : (\nabla \mu^T \nabla v^T)\} dx \\ &+ \int_{\partial D} W(v)\mu \cdot \nu d\mathcal{H}^{n-1} \end{aligned}$$

For  $p = 1, \dots, n$ , using the symmetries of  $c_{ijkl}$ ,  $\sigma_{ij}$  and  $e_{ij}$ , we have

$$\begin{aligned} \partial_p [W(v)] &= \partial_p \left[ \frac{1}{2} c_{ijkl} e_{ij} e_{kl} - f_j v_j \right] \\ &= \left[ \frac{1}{2} (\partial_p c_{ijkl}) e_{ij} e_{kl} - (\partial_p f_j) v_j \right] + \sigma_{ij} (\partial_p e_{ij}) - f_j (\partial_p v_j), \\ \sigma_{ij} (\partial_p e_{ij}) &= \sigma_{ij} \partial_p \left( \frac{\partial_i v_j + \partial_j v_i}{2} \right) \\ &= \sigma_{ij} \partial_p (\partial_i v_j) = \sigma_{ij} \partial_i (\partial_p v_j). \end{aligned}$$

Therefore, we have

$$(23) \quad \nabla_x [W(v)] = \nabla_\xi W(v) + \sigma_{ij} \partial_i (\nabla v_j) - f_j (\nabla v_j).$$

Substituting (23) into (22) and using the equality

$$\begin{aligned} \int_D \{\sigma_{ij} \partial_i (\nabla v_j)\} \cdot \mu dx &= \int_{\partial D} (\sigma[v]\nu) \cdot ((\mu \cdot \nabla)v) d\mathcal{H}^{n-1} \\ &- \int_D \operatorname{div} \sigma[v] \cdot ((\mu \cdot \nabla)v) dx, \end{aligned}$$

we obtain our objective formula.  $\square$

The following two corollaries are easily derived from this theorem. We omit their proofs.

**Corollary 4.7.** *Under the assumptions in Theorem 4.5 with (21), let  $D$  be a subdomain of  $\Omega$  with the cone property, in which the Gauss-Green formula holds. If  $u \in H^2(D, \mathbb{R}^n)$ , then we have the following formula:*

$$(24) \quad \begin{aligned} E'_*(\varphi_0)[\mu] &= \int_{\Omega \setminus D} Q[u, \mu](x) dx \\ &+ \int_{\partial D} \{W(u)\mu \cdot \nu - (\sigma[u]\nu) \cdot ((\mu \cdot \nabla)u)\} d\mathcal{H}^{n-1}, \end{aligned}$$

for  $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ .

**Corollary 4.8.** *Under the assumptions in Theorem 4.5 with (21), if there is a sequence of subdomains  $\{\Omega_l\}_l$  with the cone property in which the Gauss-Green formula holds and*

$$\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega \quad \text{with} \quad \bigcup_{l=1}^{\infty} \Omega_l = \Omega,$$

and if the global minimizer  $u$  belongs to  $H^2(\Omega_l, \mathbb{R}^n)$  for each  $l \in \mathbb{N}$ , then we have

$$(25) \quad \begin{aligned} E'_*(\varphi_0)[\mu] &= \lim_{l \rightarrow \infty} \int_{\partial \Omega_l} \{W(u)\mu \cdot \nu - (\sigma[u]\nu) \cdot ((\mu \cdot \nabla)u)\} d\mathcal{H}^{n-1}, \end{aligned}$$

for  $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ .

The above boundary integral appearing in (25) is called J-integral and (25) shows that the variation of energy is given by a limit of the J-integral in general. For more detail in a traditional setting in a cracked domain, see the next section. On the other hand, sum of the J-integral and a domain integral appearing in (24) is called generalized J-integral which was defined by Ohtsuka [17].

## 5. APPLICATION TO FRACTURE MECHANICS

In this section, we briefly illustrate an application of our theory to the energy release rate in fracture mechanics. The virtual crack extension used in the definition of the energy release rate is expressed in terms of the Lipschitz domain deformation described in Section 3.

As an important application of our results in the previous section, we consider the energy release rate  $G$  defined by (3). We set a cracked elastic body  $\Omega = \Omega_* \setminus \Sigma$  with  $\bar{\Omega} \subset \Omega_0$ , and consider its virtual crack extension  $\Omega(t) = \Omega_* \setminus \Sigma(t)$  for  $t \in [0, T]$ . We assume that the virtual crack extension is expressed by a parametrized Lipschitz deformation  $\varphi(t) \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ . More precisely, we suppose

$$(26) \quad \begin{cases} \varphi \in C^1([0, T], W^{1,\infty}(\Omega_0, \mathbb{R}^n)), \\ \varphi(t) \in \mathcal{O}'(\Omega) \quad (0 \leq t \leq T), \\ \varphi(0) = \varphi_0, \quad \Omega(t) = \Omega(\varphi(t)) \quad (0 \leq t \leq T), \\ \mathcal{H}^{n-1}(\Sigma(t) \setminus \Sigma) = t \quad (0 \leq t \leq T). \end{cases}$$

We define

$$\mu := \left. \frac{d}{dt} \varphi(t) \right|_{t=0} \in W^{1,\infty}(\Omega_0, \mathbb{R}^n).$$

In case of a virtual crack extension, without loss of generality, we can suppose

$$(27) \quad \text{supp}(\mu) \subset \Omega_*.$$

Additionally, from the inclusion relation (1), we suppose

$$(28) \quad \mu \cdot \nu = 0 \quad \text{on} \quad \Sigma,$$

where  $\nu$  is a unit normal vector on  $\Sigma$ .

The boundary conditions in (15) become as follows. Let  $\Gamma_D$  be a subportion of  $\partial\Omega_*$  with  $\mathcal{H}^{n-1}(\Gamma_D) > 0$  and define  $\Gamma_N^0 := \partial\Omega_* \setminus \Gamma_D$ . The two sides of the crack  $\Sigma$  are denoted by  $\Sigma_+$  and  $\Sigma_-$ . Then  $\Gamma_N := \partial\Omega \setminus \Gamma_D$  is divided into the following union:

$$\Gamma_N = \Gamma_N^0 \cup \Sigma_+ \cup \Sigma_-.$$

From a physical point of view, we naturally suppose that

$$(29) \quad -\sigma[u]\nu = 0 \quad \text{on} \quad \Sigma_+ \cup \Sigma_-,$$

i.e.,  $\text{supp}(h) \subset \Gamma_N^0$ .

In case of the above crack problem, the weak solution  $u$  belongs to  $H^1(\Omega)$ , but not to  $H^2(\Omega)$  in general. We remark that Theorem 4.5 is valid for all  $H^1$  weak solution without assuming additional regularity.

We consider a Lipschitz domain  $A \subset \mathbb{R}^n$  which includes all singular points of  $u$ , i.e.,  $u \in H^2(\Omega \setminus \bar{A})$ . We remark that the boundary of  $D := \Omega \setminus \bar{A}$  is decomposed into the following union:

$$\partial D = \partial\Omega_* \cup (\Sigma_+ \setminus A) \cup (\Sigma_- \setminus A) \cup ((\partial A) \cap \Omega).$$

We have following theorem for the energy release rate.

**Theorem 5.1.** *Under conditions of Theorem 4.5 with  $m \geq 1$  and the assumptions (26), the energy release rate  $G$  defined by (3) exists and is given by*

$$G = E'_*(\varphi_0)[\mu] = \int_{\Omega} Q[u, \mu] dx,$$

where  $\mu = \frac{d}{dt}\varphi(t)|_{t=0}$ . Moreover, under the conditions (27), (28) and (29), we obtain

$$G = \int_{\Omega \cap A} Q[u, \mu] dx + \int_{(\partial A) \cap \Omega} \{W(u)\mu \cdot \nu - (\sigma[u]\nu) \cdot ((\mu \cdot \nabla)u)\} d\mathcal{H}^{n-1}.$$

*Proof.* The first part is a direct consequence of Theorem 4.5. From the conditions (27), (28) and (29), the latter part is derived from Corollary 4.7, too.  $\square$

If  $\Sigma$  is a straight crack in a two dimensional domain, then we can choose  $\mu$  as it is constant vector around a crack tip. In other words, we can assume that

$$(30) \quad \nabla\mu^T = 0 \quad \text{in} \quad A.$$

Even in higher dimension, if  $\Sigma$  is flat and the crack extension is a translation, we can assume the condition (30). In this case, we have the following corollary.

**Corollary 5.2.** *Under the additional condition (30), the formula:*

$$(31) \quad G = \int_{(\partial A) \cap \Omega} \{W(u)\mu \cdot \nu - (\sigma[u]\nu) \cdot ((\mu \cdot \nabla)u)\} d\mathcal{H}^{n-1}$$

holds, where the above boundary integral on  $(\partial A) \cap \Omega$  is path-independent.

The path-independent integral in (31) is called the J-integral.

## REFERENCES

- [1] Adams, R. A.: *Sobolev Spaces*, Academic Press, New York-London, 1975.
- [2] Berger, M. S.: *Nonlinearity and Functional Analysis*, Academic Press, New York-London, 1977.
- [3] Bourdin, B., Francfort, G. A., and Marigo, J.-J.: The variational approach to fracture, *J. Elasticity* **91** (2008) 5–148.
- [4] Cherepanov, G. P.: On crack propagation in continuous media, *Prikl. Math. Mekh.* **31**, No.3 (1967) 476–493.
- [5] Evans, L. C., and Gariepy, R. F.: *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
- [6] Francfort, G. A., and Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem, *J. Mech. Phys. Solids* **46** (1998) 1319–1342.
- [7] Friedit, G., and Sokolowski, J.: Hadamard formula in nonsmooth domains and applications, *Partial differential equations on multistructures (Luminy, 1999)*, *Lecture Notes in Pure and Appl. Math.* **219** Dekker, New York (2001) 99–120.
- [8] Friedit, G., and Sokolowski, J.: Shape sensitivity analysis of problems with singularities, *Shape optimization and optimal design (Cambridge, 1999)*, *Lecture Notes in Pure and Appl. Math.* **216** Dekker, New York (2001) 255–276.
- [9] Griffith, A. A.: The phenomenon of rupture and flow in solids, *Phil. Trans. Royal Soc. London* **A221** (1920) 163–198.
- [10] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations*, *Lecture Notes in Mathematics* **840**, Springer Verlag, 1981.
- [11] Horgan, C. O.: Korn's inequalities and their applications in continuum mechanics, *SIAM Rev.* **37** No.4 (1995) 491–511.
- [12] Irwin, G. R.: Fracture dynamics, in: *Fracturing of Metals* (1948) 147–166, American Society of Metals.

- [13] Ito, H.: Extended Korn's inequalities and the associated best possible constants, *J. Elasticity* **24** (1990) 43-78.
- [14] Kimura, M.: Shape derivative and linearization of moving boundary problems, *GAKUTO International Series, Mathematical Sciences and Applications* **13** (2000) 167-179.
- [15] Kimura, M. and Wakano, I.: New mathematical approach to the energy release rate in crack extension, *Trans. Japan Soc. Indust. Appl. Math.* **16**, No.3 (2006) 345-358. (in Japanese)
- [16] Kovtunenkov, V. A.: Primal-dual methods of shape sensitivity analysis for curvilinear cracks with nonpenetration, *IMA J. Appl. Math.* **71**, No.5 (2006) 635-657.
- [17] Ohtsuka, K.: Generalized J-integral and three-dimensional fracture mechanics I., *Hiroshima Math. J.* **11** (1981) 21-52.
- [18] Ohtsuka, K.: Generalized J-integral and its applications I – Basic theory –, *Japan J. Appl. Math.* **2**, No.2 (1985) 329-350.
- [19] Ohtsuka, K.: Generalized J-integral and three-dimensional fracture mechanics II, *Hiroshima Math. J.* **16** (1986) 327-352.
- [20] Ohtsuka, K., and Khludnev, A.: Generalized J-integral method for sensitivity analysis of static shape design, *Control & Cybernetics* **29** (2000) 513-533.
- [21] Rice, J. R.: A path-independent integral and the approximate analysis of strain concentration by notches and cracks, *J. Appl. Mech.* **35** (1968) 379-386.
- [22] Sokolowski, J., and Zolesio, J.-P.: *Introduction to Shape Optimization, Shape Sensitivity Analysis*, Springer Ser. Comput. Math. **16**, Springer-Verlag, Berlin, Heidelberg, 1992.
- [23] Ziemer, W. P.: *Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation*, Graduate Texts in Mathematics **120**, Springer-Verlag, New York, 1989.

Masato Kimura  
Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan  
E-mail: masato(at)math.kyushu-u.ac.jp

Isao Wakano  
Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan  
E-mail: wakano(at)i.kyoto-u.ac.jp