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Exact computation for the cover times of certain classes of trees

Yusuke Higuchi, Takuya Ohwa and Tomoyuki Shirai

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Abstract. We show a special feature for the cover time of trees that is not satisfied by those of other graphs. By using this property, we show the relationship between the cover times of a tree and its subdivision, and we compute exactly the distribution of the last vertex visited by a random walk, the expectation and the Laplace transform of cover times of spider graphs as integral representations. We also discuss some comparison results for spider graphs.

Keywords. Cover time, first hitting time, terminal time, tree, subdivision, spider graph.

1. Introduction

Let $G = (V, E)$ be a finite connected graph possibly with self-loops. Throughout this paper, we consider a discrete-time, irreducible random walk $(\{X_t\}_{t\in\mathbb{Z}_{\geq 0}}, \{\mathbb{P}_x\}_{x\in V})$ on G with transition matrix $P = (p(x, y))_{x,y\in V}$ where $\mathbb{Z}_{\geq 0} = \mathbb{Z} \cap [0, \infty)$. We suppose that P is compatible with the graph structure in the sense that $p(x, y), p(y, x) > 0$ if and only if $xy \in E$. The cover time $C = C_G$ is the number of steps needed for a random walk to visit all the vertices of G. In other words,

$$C = \max_{x\in V} \sigma_x \tag{1.1}$$

with σ_x being the first hitting time to a vertex $x \in V$. The basic results for cover times can be found, for instance, in [1, 3] and references therein. In [4, 5], we obtained the Möbius inversion formula for the cover time in terms of the first exit times.

Theorem 1.1 ([4, 5]). *Let $\mathcal{C}(V)$ be the totality of vertex sets of connected subgraphs of G, and set*

$$\mathcal{D}_x = \{A \in \mathcal{C}(V); N(A) = V, A \neq V, A \ni x\},$$

where $N(A)$ is the 1-neighborhood of a set A, i.e.,

$$N(A) = A \cup \bigcup_{x\in A} N_x$$

and N_x is the neighborhood of $x \in V$. Then, for $x, y \in V$ and $t \in \mathbb{Z}_{\geq 0}$, we have

$$\mathbb{P}_x(C = t, X_C = y) = \sum_{B\in\mathcal{D}_x} (-1)^{N-|B|+1}\mathbb{P}_x(\tau_B = t, X_{\tau_B} = y),$$

where $\tau_B = \inf\{t \in \mathbb{Z}_{\geq 0}; X_t \notin B\}$ is the first exit time from a subset B. Moreover, the set $\mathcal{D}_x$ in the formula can be replaced by $\mathcal{D}_x \setminus \mathcal{D}_y$.

From this theorem, once we characterize the sets $\mathcal{D}_x$ and $\mathcal{D}_y$, we can write down the joint distribution of the cover time and the last visited point as an alternating sum.

The situation becomes simpler for random walks on trees. In what follows, we only consider the case where G is a tree. We say that a vertex l of a tree is a *leaf* if its degree is one; the set of all leaves is denoted by L. In the case of trees, we can rephrase Theorem 1.1 as follows:

Theorem 1.2. *Let G be a finite tree and L the set of its leaves. Then, for $x, y \in V$ and $t \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathbb{P}_x(C = t, X_C = y) = \sum_{\substack{\Lambda\subset L\\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|+1}\mathbb{P}_x(\sigma_\Lambda = t, X_{\sigma_\Lambda} = y).$$

In particular, for any function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_x[f(C)] = \sum_{\substack{\Lambda\subset L\\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|+1}\mathbb{E}_x[f(\sigma_\Lambda)] \tag{1.2}$$

where $\sigma_\Lambda = \inf\{t \in \mathbb{Z}_{\geq 0}; X_t \in \Lambda\}$ is the first hitting time to a (non-empty) subset $\Lambda \subset L$. Similarly,

$$\mathbb{P}_x(X_C = y) = \sum_{\substack{\Lambda\subset L\\ \Lambda\ni y}} (-1)^{|\Lambda|+1}\mathbb{P}_x(X_{\sigma_\Lambda} = y). \tag{1.3}$$

Remark 1.3. From (1.3) we see that $\mathbb{P}_x(X_C = y) = 0$ unless $y \in L$, that is, the maximum of first hitting times is attained at a certain leaf, which implies that when G is a tree, (1.1) can be written as $C = \max_{l\in L} \sigma_l$. On the other hand, $\sigma_\Lambda = \min_{l\in\Lambda} \sigma_l$ for $\Lambda \subset L$. Therefore, the formula above can be understood as the ordinary inclusion-exclusion principle when G is a tree.

In Section 2, we remark that the cover times of trees have similar property to what is enjoyed by the first hitting times (Lemmas 2.1 and 2.2). We also derive the formula clarifying the relationship between the cover times of a tree and its subdivisions (Theorem 2.5).

In Section 3, by using Theorem 1.2, we will give integral representations for the Laplace transform of cover times of spider graphs, the distribution of the last vertex visited by a random walk and the expectation (Theorems 3.1, 3.7, and 3.10). As a corollary, we remark some comparison results for the cover time of spider graphs (Corollary 3.8 and Remarks 3.11).

2. Cover time of the n-subdivision of a tree

Let $W = \{w : \mathbb{Z}_{\geq 0} \to V\}$ be the set of discrete-time random walk paths on a set V. Let $\mathcal{F}_t$ be the filtration generated by the coordinate maps $\{X_s(w) = w(s), s = 0, 1, 2, \ldots, t\}$. An $\{\mathcal{F}_t\}$-stopping time $\tau : W \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a *terminal time* under $\mathbb{P}_x$ if it satisfies

$$\tau(w) = k + \tau(\theta_k w), \quad \mathbb{P}_x\text{-a.s. on } \{\tau \geq k\},$$

for all $k \in \mathbb{Z}_{\geq 0}$, where $(\theta_k w)(n) = w(n+k), n \in \mathbb{Z}_{\geq 0}$ is the shifted path by time $k \in \mathbb{Z}_{\geq 0}$ (cf. [2]). The first hitting time σ_A to a set A is a terminal time under $\mathbb{P}_x$ for $x \in V$ while the cover time is not since the cover time needs memory of which vertices are visited. Nevertheless the following lemma holds.

Lemma 2.1. *The cover time on a tree satisfies*

$$C(w) = k + C(\theta_k w), \quad \mathbb{P}_x\text{-a.s. on } \{\sigma_L \geq k\},$$

for $x \in V$ and $k \in \mathbb{Z}_{\geq 0}$. In particular,

$$C(w) = 1 + C(\theta_1 w) \tag{2.1}$$

under $\mathbb{P}_x$ for $x \in V \setminus L$.

Proof. From Remark 1.3, we see that

$$C(w) = \max_{l \in L} \sigma_l(w) = \max_{l \in L}(k + \sigma_l(\theta_k w)) = k + C(\theta_k w),$$

$\mathbb{P}_x$-a.s. on $\{\sigma_L \geq k\}$ for every $x \in V$. Here we used the fact that σ_l is a terminal time and $\sigma_L \geq k$ means that $\sigma_l \geq k$ for all $l \in L$. □

This is a remarkable feature for the cover times of trees and it often makes the situation simpler.

When G is a tree, the cover times inherit some properties such as harmonicity from the first hitting times by Lemma 2.1.

Lemma 2.2. *Let G be a tree and $l \in L$ be a leaf.*
(1) The function $\psi_0(x) = \mathbb{P}_x(X_C = l)$ is P-harmonic in $V \setminus L$, i.e.,

$$\psi_0(x) = (P\psi_0)(x) \tag{2.2}$$

for $x \in V \setminus L$, where $(Pf)(x) = \sum_{y \in V} p(x,y) f(y)$. In particular, $\psi_0(x)$ is determined as the unique harmonic extension if the boundary values $\{\psi_0(l), l \in L\}$ are known.
(2) The function $\psi_1(x) = \mathbb{E}_x[C]$ satisfies

$$\psi_1(x) = 1 + (P\psi_1)(x) \tag{2.3}$$

for $x \in V \setminus L$.

Proof. Both assertions follow immediately from (2.1). They also follow from Theorem 1.2 since $\phi_{\Lambda,0}(x) = \mathbb{P}_x(X_{\sigma_\Lambda} = l)$ and $\phi_{\Lambda,1}(x) = \mathbb{E}_x[\sigma_\Lambda]$ satisfy the same equations (2.2) and (2.3), respectively. □

Remark 2.3. The cover times for general graphs with cycles do not satisfy (2.2) and (2.3).

Now we introduce the notion of the n-subdivision of a graph.

Definition 2.4. For a graph $G = (V, E)$, the n-subdivision $G_n = (V_n, E_n)$ of G is defined by replacing each edge in E with a path of length $n+1$. The vertex set V is naturally regarded as a subset of V_n, and we write $V \subset V_n$.

It is worthwhile to notice a trivial fact that the set of leaves of a tree T leaves unchanged by the operation of taking n-subdivision. So we can identify the set of leaves in V with that in V_n in an obvious manner.

The next theorem shows that the simple relationship between cover times on a tree and its subdivision.

Theorem 2.5. *Let $T = (V, E)$ be a tree and $T_{n-1} = (V_{n-1}, E_{n-1})$ its $(n-1)$-subdivision for $n \geq 2$. We denote by $(\{X_t\}_{t \in \mathbb{Z}_{\geq 0}}, \{\mathbb{P}_x\}_{x \in V})$ and $(\{\tilde{X}_t\}_{t \in \mathbb{Z}_{\geq 0}}, \{\tilde{\mathbb{P}}_x\}_{x \in V_{n-1}})$ the simple random walks on T and T_{n-1}, respectively. Then, when $y \in V_{n-1}$ is a vertex between a and b for an edge $ab \in E$, for $l \in L$,*

$$\tilde{\mathbb{P}}_y(\tilde{X}_{C_{T_{n-1}}} = l) = (1 - \frac{k}{n})\mathbb{P}_a(X_{C_T} = l) + \frac{k}{n}\mathbb{P}_b(X_{C_T} = l) \tag{2.4}$$

and

$$\tilde{\mathbb{E}}_y[C_{T_{n-1}}] = k(n-k) + n^2 \left\{ (1 - \frac{k}{n})\mathbb{E}_a[C_T] + \frac{k}{n}\mathbb{E}_b[C_T] \right\}, \tag{2.5}$$

where $k = d(a, y)$ is the (shortest path) distance between a and y in T_{n-1}. In particular, if $a \in V \subset V_{n-1}$,

$$\tilde{\mathbb{P}}_a(\tilde{X}_{C_{T_{n-1}}} = l) = \mathbb{P}_a(X_{C_T} = l), \tag{2.6}$$

$$\tilde{\mathbb{E}}_a[C_{T_{n-1}}] = n^2 \mathbb{E}_a[C_T]. \tag{2.7}$$

Proof. Let us consider an edge $ab \in E$ and the corresponding path P_{ab} of length n in T_{n-1} with end-points a and b. Since the function $\psi_0(y) = \tilde{\mathbb{P}}_y(\tilde{X}_{C_{T_{n-1}}} = l)$ for a leaf $l \in L$ is harmonic outside L from Lemma 2.2(1), the values $\psi_0(y)$ on the path P_{ab} is determined as a linear function with boundary conditions $\psi_0(a)$ and $\psi_0(b)$ at a and b, respectively. Similarly, the function $\psi_1(y) = \tilde{\mathbb{E}}_x[C_{T_{n-1}}]$ satisfies (2.3). Hence, it is easy to see that

$$\psi_1(y) = k(n-k) + (1 - \frac{k}{n})\psi_1(a) + \frac{k}{n}\psi_1(b). \tag{2.8}$$

Therefore, (2.4) and (2.5) follow from (2.6) and (2.7), respectively.

For (2.6) and (2.7), it suffices to show that the same equations hold for the first hitting time σ_Λ to each nonempty subset $\Lambda \subset L$ because of (1.2) and (1.3), which follow from Lemma 2.6. □

Lemma 2.6. *Let T_{n-1} be the $(n-1)$-subdivision of a tree T. Let Λ be a non-empty subset of L and $l \in L$. Then,*

$$\tilde{\mathbb{P}}_a(X_{\sigma_\Lambda} = l) = \mathbb{P}_a(X_{\sigma_\Lambda} = l), \tag{2.9}$$

$$\tilde{\mathbb{E}}_a[\sigma_\Lambda] = n^2 \mathbb{E}_a[\sigma_\Lambda]. \tag{2.10}$$

for $a \in V \subset V_{n-1}$.

Proof. We only show (2.10). The equality (2.9) can be shown in the same manner. Let $\tilde{\phi}_{1,\Lambda}(x) = \tilde{E}_x[\sigma_\Lambda]$ for $x \in V_{n-1}$. It holds that

$$\tilde{\phi}_{1,\Lambda}(a) = 1 + \frac{1}{|N_a(T_{n-1})|} \sum_{y \in N_a(T_{n-1})} \tilde{\phi}_{1,\Lambda}(y), \tag{2.11}$$

for $a \in V \subset V_{n-1}$, where $N_a(T_{n-1})$ is the neighborhood of the vertex a in T_{n-1}. By the same argument as before, we see that

$$\tilde{\phi}_{1,\Lambda}(y) = n - 1 + (1 - \frac{1}{n})\tilde{\phi}_{1,\Lambda}(a) + \frac{1}{n}\tilde{\phi}_{1,\Lambda}(b) \tag{2.12}$$

when $y \in N_a(T_{n-1})$ is between a and b for $ab \in E$. By substituting (2.12) into (2.11), we have

$$\frac{\tilde{\phi}_{1,\Lambda}(a)}{n^2} = 1 + \frac{1}{|N_a(T)|} \sum_{y \in N_a(T)} \frac{\tilde{\phi}_{1,\Lambda}(y)}{n^2}.$$

where $N_a(T)$ is the neighborhood of a vertex $a \in V$ in T, and $|N_a(T_{n-1})| = |N_a(T)|$. Hence, $\{\frac{\tilde{\phi}_{1,\Lambda}(a)}{n^2},\ a \in V \subset V_n\}$ satisfies the same equation as $\{\phi_{1,\Lambda}(x) := \mathbb{E}_x[\sigma_\Lambda], x \in V\}$ should satisfy. In the same manner, $\frac{\tilde{\phi}_{1,\Lambda}(l)}{n^2}$ for $l \in L$ satisfies the same boundary condition as that for $\phi_{1,\Lambda}(l)$. Consequently, we can conclude that $\tilde{\phi}_{1,\Lambda}(a) = n^2\phi_{1,\Lambda}(a)$ for any $a \in V \subset V_{n-1}$. □

3. Cover times of spider graphs

Let $G_{m_1,m_2,\ldots,m_K}$ be a spider graph which is obtained from K paths such that each length is $m_i, i \in [K] := \{1, 2, \ldots, K\}$ by identifying each one of the end-points. We call the identified point the origin and denote it by 0. More precisely, the vertex set and the edge set are written as

$$V = \{0\} \cup \bigcup_{i=1}^{K} \{(i, x), x = 1, 2, \ldots, m_i\},$$

$$E = \bigcup_{i=1}^{K} \{(i, x-1)(i, x), x = 1, 2, \ldots, m_i\},$$

where $(i, 0)$ is understood as the origin 0. We denote the leaf (i, m_i) by l_i for $i \in [K]$. Throughout this section, for simplicity, we consider *simple* random walks on spider graphs.

We will compute the Laplace transform of the cover time of a spider graph.

Theorem 3.1. *The Laplace transform of the cover time of a spider graph $G_{m_1,\ldots,m_K}$ by a simple random walk is given by*

$$\mathbb{E}_0[e^{-\lambda C}] = \sum_{k=1}^{K} \frac{1}{b_k(\lambda)} \int_0^\infty e^{-a_k^{-1}t} \prod_{\substack{j=1 \\ j \neq k}}^{K} (e^{-a_j t} - e^{-a_j^{-1} t}) dt$$

where $a_k(\lambda) = \tanh(m_k \kappa(\lambda))$ and $b_k(\lambda) = \sinh(m_k \kappa(\lambda))$ with $\kappa(\lambda) = \log(e^\lambda + \sqrt{e^{2\lambda} - 1})$.

Remark 3.2. Given $m_i > 0, i \in [K]$, let C_n be the cover time of $G_{nm_1,\ldots,nm_K}$. Since $n\kappa(\lambda/n^2) \to \sqrt{2\lambda}$ as $n \to \infty$, it is easy to verify that

$$\lim_{n\to\infty} \mathbb{E}_0[e^{-\lambda C_n/n^2}] = \sum_{k=1}^{K} \frac{1}{B_k(\lambda)} \int_0^\infty e^{-A_k^{-1}t} \prod_{\substack{j=1 \\ j \neq k}}^{K} (e^{-A_j t} - e^{-A_j^{-1} t}) dt,$$

where $A_k(\lambda) = \tanh(m_k\sqrt{2\lambda})$ and $B_k(\lambda) = \sinh(m_k\sqrt{2\lambda})$. It should be the Laplace transform of a constant multiple of the cover time of K-rays by Walsh's Brownian motion starting at the origin (the common vertex). Similar expressions can be found in Section 17.2.3, [7]. So far we do not know the inverse Laplace transform.

Before computing the Laplace transform, we compute $\mathbb{P}_x(w_C = l_j)$ and $\mathbb{E}_x[C]$. First we show three lemmas. In what follows, we set

$$\alpha_j = m_j^{-1}, \quad j \in [K]$$

and use the following notation

$$[\Lambda] = \{k \in [K]; l_k \in \Lambda\}$$

for $\Lambda \subset L$.

Lemma 3.3. *For a given $j \in [K]$, let $\Lambda \subset L$ with $\Lambda \ni l_j$. Then,*

$$\mathbb{P}_0(X_{\sigma_\Lambda} = l_j) = \frac{\alpha_j}{\sum_{k\in[\Lambda]} \alpha_k} \tag{3.1}$$

$$\mathbb{P}_{l_k}(X_{\sigma_\Lambda} = l_j) = \begin{cases} \delta_{k,j}, & k \in [\Lambda], \\ \mathbb{P}_0(X_{\sigma_\Lambda} = l_j), & k \notin [\Lambda]. \end{cases} \tag{3.2}$$

Proof. Set $\phi_\Lambda(k, x) = \mathbb{P}_{(k,x)}(X_{\sigma_\Lambda} = l_j)$ for $x \in [m_k]$ and $\phi_\Lambda(0) = \mathbb{P}_0(X_{\sigma_\Lambda} = l_j)$. Since ϕ_Λ satisfies (2.2), we may assume that

$$\phi_\Lambda(k, x) = \beta_\Lambda(k)x + \gamma_\Lambda, \quad k \in [K],$$

where $\gamma_\Lambda = \phi_\Lambda(0)$. They satisfy the following boundary conditions:

$$\begin{cases} \phi_\Lambda(k, m_k) = \delta_{k,j}, & k \in [\Lambda], \\ \phi_\Lambda(k, m_k) = \phi_\Lambda(k, m_k - 1), & k \in [K] \setminus [\Lambda], \\ \phi_\Lambda(0) = \frac{1}{K} \sum_{k\in[K]} \phi_\Lambda(k, 1). \end{cases}$$

A simple computation yields

$$\begin{cases} \beta_\Lambda(k) = \alpha_k(\delta_{k,j} - \gamma_\Lambda), & k \in [\Lambda] \\ \beta_\Lambda(k) = 0, & k \in [K] \setminus [\Lambda], \\ \sum_{k\in[K]} \beta_\Lambda(k) = 0. \end{cases}$$

Hence, we obtain

$$\gamma_\Lambda = \frac{\alpha_j}{\sum_{k\in[\Lambda]} \alpha_k}.$$

Therefore, we obtain (3.1) and (3.2). □

Lemma 3.4. *Let $\Lambda \subset L$ with $\Lambda \neq \emptyset$. Then,*

$$\mathbb{E}_0[\sigma_\Lambda] = \frac{2\sum_{k\in[K]} m_k - \sum_{k\in[\Lambda]} m_k}{\sum_{k\in[\Lambda]} \alpha_k}. \tag{3.3}$$

Proof. Set $\phi_\Lambda(k,x) = \mathbb{E}_{(k,x)}[\sigma_\Lambda]$ for $x \in [m_k]$ and $\phi_\Lambda(0) = \mathbb{E}_0[\sigma_\Lambda]$. Since ϕ_Λ satisfies (2.3), we may assume that

$$\phi_\Lambda(k,x) = -x^2 + \beta_\Lambda(k)x + \gamma_\Lambda, \tag{3.4}$$

where $\gamma_\Lambda = \phi_\Lambda(0)$. They satisfy the following boundary conditions:

$$\begin{cases} \phi_\Lambda(k, m_k) = 0, & l_k \in \Lambda, \\ \phi_\Lambda(k, m_k) = 1 + \phi_\Lambda(k, m_k - 1), & l_k \in \Lambda^c, \\ \phi_\Lambda(0) = 1 + \frac{1}{K}\sum_{k\in[K]} \phi_\Lambda(k,1). \end{cases}$$

A simple computation yields

$$\begin{cases} \beta_\Lambda(k) = m_k - \alpha_k\gamma_\Lambda, & l_k \in \Lambda, \\ \beta_\Lambda(k) = 2m_k, & l_k \in \Lambda^c, \\ \sum_{k\in[K]} \beta_\Lambda(k) = 0. \end{cases} \tag{3.5}$$

Hence,

$$\gamma_\Lambda = \frac{2\sum_{k\in[K]} m_k - \sum_{k\in[\Lambda]} m_k}{\sum_{k\in[\Lambda]} \alpha_k}.$$

□

Lemma 3.5. *Let S be a finite set, and let μ_1 and μ_2 be positive measures and ν a signed measure on S. Suppose that μ_1 has full support. Then, for $z \in \mathbb{C}$, we have*

$$\sum_{\substack{\Lambda\subset S\\ \Lambda\neq\emptyset}} \frac{z^{|\Lambda|}}{\mu_1(\Lambda)} = \int_0^\infty \left\{ \prod_{y\in S}(1 + ze^{-\mu_1(y)t}) - 1 \right\} dt$$

and

$$\sum_{\substack{\Lambda\subset S\\ \Lambda\neq\emptyset}} \frac{\nu(\Lambda)}{\mu_1(\Lambda) + \mu_2(\Lambda^c)} z^{|\Lambda|} = \sum_{x\in S} \nu(x) \int_0^\infty ze^{-\mu_1(x)t} \prod_{y\in S\setminus\{x\}} (ze^{-\mu_1(y)t} + e^{-\mu_2(y)t})dt.$$

In particular,

$$\sum_{\substack{\Lambda\subset S\\ \Lambda\ni x}} \frac{z^{|\Lambda|}}{\mu_1(\Lambda)} = \int_0^\infty ze^{-\mu_1(x)t} \prod_{y\in S\setminus\{x\}} (1 + ze^{-\mu_1(y)t})dt$$

Proof. We only show the second formula.

$$\begin{aligned} &\sum_{\substack{\Lambda\subset S\\ \Lambda\neq\emptyset}} \frac{\nu(\Lambda)}{\mu_1(\Lambda) + \mu_2(\Lambda^c)} z^{|\Lambda|} \\ &= \sum_{\substack{\Lambda\subset S\\ \Lambda\neq\emptyset}} \sum_{x\in\Lambda} \nu(x) z^{|\Lambda|} \int_0^\infty e^{-\{\mu_1(\Lambda)+\mu_2(\Lambda^c)\}t} dt \\ &= \sum_{x\in S} \nu(x) \sum_{\Lambda\ni x} z^{|\Lambda|} \int_0^\infty e^{-\mu_1(\Lambda)t-\mu_2(\Lambda^c)t} dt \\ &= \sum_{x\in S} \nu(x) \int_0^\infty ze^{-\mu_1(x)t} \prod_{y\in S\setminus\{x\}} (ze^{-\mu_1(y)t} + e^{-\mu_2(y)t})dt. \end{aligned}$$

The first formula is obtained in the same manner. □

Remark 3.6. In the same manner as before, we can show the following. Let S be a finite set with $|S| = n$ and μ a positive measure with full support. Then, we can see that

$$\sum_{\substack{\Lambda\subset S\\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|+1} \mu(\Lambda)^{-1} = \mathbb{E}[\max_{i\in S} X_i]$$

and

$$\sum_{\substack{\Lambda\subset S\\ \Lambda\ni x}} (-1)^{|\Lambda|+1} \mu(\Lambda \mid \{x\})^{-1} = \mathbb{P}(\max_{i\in S} X_i = X_x),$$

where $\{X_i, i \in S\}$ are the i.i.d. random variables such that each X_i is exponential with parameter $\mu(\{i\})$ respectively.

We define

$$\Phi_I(t) = \prod_{k\in[K]\setminus I} (1 - e^{-\alpha_k t})$$

for $I \subset [K]$. For example, when $I = \{1, 2, 3\}$, we simply write $\Phi_{1,2,3}(t)$. We note that $0 < \Phi_I(t) < 1$ for all $t > 0$ unless $I = [K]$.

Theorem 3.7. *Let C be the cover time of a spider graph $G_{m_1,m_2,\dots,m_K}$ by a simple random walk. Then,*

$$\mathbb{P}_0(X_C = l_j) = \int_0^\infty \alpha_j e^{-\alpha_j t} \Phi_j(t)dt$$

$$\mathbb{P}_{l_k}(X_C = l_j) = (1 - \delta_{k,j}) \int_0^\infty \alpha_j e^{-\alpha_j t} \Phi_{j,k}(t)dt$$

for each leaf $l_j \in L$. Moreover,

$$\begin{aligned} &\mathbb{P}_{(k,x)}(X_C = l_j) \\ &= \mathbb{P}_{l_k}(X_C = l_j)\alpha_k x + \mathbb{P}_0(X_C = l_j)(1 - \alpha_k x). \end{aligned}$$

Proof. From (1.3), (3.1) and Lemma 3.5, we obtain

$$\begin{aligned} \mathbb{P}_0(X_C = l_j) &= \sum_{\substack{\Lambda\subset L\\ \Lambda\ni l_j}} (-1)^{|\Lambda|+1} \mathbb{P}_0(X_{\sigma_\Lambda} = l_j) \\ &= \sum_{\substack{\Lambda\subset L\\ \Lambda\ni l_j}} (-1)^{|\Lambda|+1} \frac{\alpha_j}{\sum_{k\in[\Lambda]} \alpha_k} \\ &= \int_0^\infty \alpha_j e^{-\alpha_j t} \Phi_j(t)dt. \end{aligned}$$

We also obtain the second formula in the same manner by noticing that the alternating sum in (1.3) should be taken over the set $\{\Lambda \subset L\setminus\{l_k\}; \Lambda \ni l_j\}$ if $k \neq j$ because of (3.2). The last assertion immediately follows from harmonicity. □

Corollary 3.8. *Suppose $m_i \le m_j$. Then,*

$$\mathbb{P}_0(X_C = l_i) \le \mathbb{P}_0(X_C = l_j). \tag{3.6}$$

Proof. Suppose $m_i \le m_j$, or equivalently $\alpha_i \ge \alpha_j$. Then,

$$\begin{aligned}
&\mathbb{P}_0(X_C = l_j) - \mathbb{P}_0(X_C = l_i)\\
&= \int_0^\infty \{\alpha_j e^{-\alpha_j t}\Phi_j(t) - \alpha_i e^{-\alpha_i t}\Phi_i(t)\}dt\\
&= \int_0^\infty \{\alpha_j e^{-\alpha_j t}(1-e^{-\alpha_i t}) - \alpha_i e^{-\alpha_i t}(1-e^{-\alpha_j t})\}\Phi_{i,j}(t)dt\\
&= \alpha_i\alpha_j \int_0^\infty dt \int_0^t ds e^{-\alpha_j t - \alpha_i s}\{1 - e^{-(\alpha_i-\alpha_j)(t-s)}\}\Phi_{i,j}(t)\\
&\ge 0.
\end{aligned}$$

□

Remark 3.9. It follows from Lemma 3.3 that

$$\mathbb{P}_0(X_{\sigma_L} = l_j) = \frac{\alpha_j}{\sum_{k=1}^K \alpha_k}.$$

Hence, as expected, the converse inequality to (3.6) holds, i.e., when $m_i \le m_j$, then

$$\mathbb{P}_0(X_{\sigma_L} = l_i) \ge \mathbb{P}_0(X_{\sigma_L} = l_j).$$

Theorem 3.10. *Let C be the cover time of a spider graph $G_{m_1,m_2,\dots,m_K}$ by a simple random walk. Then,*

$$\begin{aligned}
\mathbb{E}_0[C] &= \sum_{k\in[K]} m_k \int_0^\infty \{2 - (2-e^{-\alpha_k t})\Phi_k(t)\}dt\\
\mathbb{E}_{l_j}[C] &= \sum_{k\in[K]} m_k \int_0^\infty \{2 - (2-e^{-\alpha_k t})\phi_k^{(j)}(t)\}dt\\
&\quad + 2m_j \int_0^\infty (1-\Phi_j(t))dt
\end{aligned}$$

for every leaf $l_j \in L$, where

$$\phi_k^{(j)}(t) = \begin{cases} \Phi_{j,k}(t), & k \neq j,\\ 1, & k = j.\end{cases}$$

Proof. From (1.2), (3.3) and Lemma 3.5, we obtain

$$\begin{aligned}
\mathbb{E}_0[C] &= \sum_{\substack{\Lambda\subset L\\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|+1} \frac{2\sum_{k\in[K]} m_k - \sum_{k\in[\Lambda]} m_k}{\sum_{k\in[\Lambda]}\alpha_k}\\
&= 2\Big(\sum_{k\in[K]} m_k\Big)\int_0^\infty (1-\Phi_\emptyset(t))dt\\
&\quad - \sum_{k\in[K]} m_k \int_0^\infty e^{-\alpha_k t}\Phi_k(t)dt\\
&= \sum_{k\in[K]} m_k \int_0^\infty \{2-(2-e^{-\alpha_k t})\Phi_k(t)\}dt.
\end{aligned}$$

For $\mathbb{E}_{l_j}[C]$, we notice that

$$\mathbb{E}_{l_j}[\sigma_\Lambda] = \begin{cases} m_j^2 + \mathbb{E}_0[\sigma_\Lambda], & \text{if } \Lambda \not\ni l_j,\\ 0, & \text{if } \Lambda \ni l_j,\end{cases}$$

from (3.4) and (3.5). Then, we see that

$$\begin{aligned}
\mathbb{E}_{l_j}[C] &= \sum_{\substack{\Lambda\subset L\\ \Lambda\not\ni l_j,\Lambda\neq\emptyset}} (-1)^{|\Lambda|+1}(m_j^2 + \mathbb{E}_0[\sigma_\Lambda])\\
&= m_j^2 + \sum_{\substack{\Lambda\subset L\\ \Lambda\not\ni l_j,\Lambda\neq\emptyset}} (-1)^{|\Lambda|+1}\frac{2\sum_{k\in[K]} m_k - \sum_{k\in[\Lambda]} m_k}{\sum_{k\in[\Lambda]}\alpha_k}\\
&= m_j^2 + 2\Big(\sum_{k\in[K]} m_k\Big)\int_0^\infty (1-\Phi_j(t))dt\\
&\quad - \sum_{k\in[K]\setminus\{j\}} m_k \int_0^\infty e^{-\alpha_k t}\Phi_{j,k}(t)dt\\
&= m_j^2 + 2m_j\int_0^\infty (1-\Phi_j(t))dt\\
&\quad + \sum_{k\in[K]\setminus\{j\}} m_k \int_0^\infty \{2-(2-e^{-\alpha_k t})\Phi_{j,k}(t)\}dt
\end{aligned}$$

□

Remark 3.11. It is obvious that for any tree $\mathbb{E}_x[C]$ attains the minimum at a certain leaf. When a spider graph with $K = 3, 4$, it is not difficult to see that $\mathbb{E}_{l_i}[C] \le \mathbb{E}_{l_j}[C]$ if $m_i \ge m_j$. However, when $K \ge 5$, it is not necessarily the case. Indeed, for $G_{1,2,11,13,13}$, $\mathbb{E}_{l_2}[C] \ge \mathbb{E}_{l_1}[C]$ even though $m_2 \ge m_1$. Here the spider graph $G_{1,2,11,13,13}$ is minimal among such graphs in the lexicographic order when $K = 5$. We feel that $\mathbb{E}_l[C]$ attains the minimum at the leaf ℓ_i corresponding to the largest m_i.

Example 3.12. We consider the regular spider $G_{m_1,\dots,m_K}$ with $m_i = N/K \in \mathbb{N}$, $i = 1, 2, \dots, K$. From Theorem 3.10, we see that

$$\begin{aligned}
\mathbb{E}_0[C] &= K\frac{N}{K}\int_0^\infty \{2-(2-e^{-Kt/N})(1-e^{-Kt/N})^{K-1}\}dt\\
&= \frac{N^2}{K}\left(2\sum_{k=1}^K \frac{1}{k} - \frac{1}{K}\right).
\end{aligned}$$

It also follows from Theorem 2.5 since $G_{m_1,\dots,m_K}$ with $m_i = N/K$ is the $(N/K-1)$-subdivision of $G_{1,\dots,1}$. If $K = N^\alpha (0<\alpha<1)$,

$$\mathbb{E}_0[C] \sim 2\alpha N^{2-\alpha}\log N \quad (N\to\infty).$$

Lemma 3.13. *Given a non-empty set $\Lambda \subset L$ and $\gamma_i \in \mathbb{R}$ for $i \in [\Lambda]$. Let ϕ_Λ be a function on the spider graph $G_{m_1,\dots,m_K} = (V, E)$ satisfying for $\lambda > 0$*

$$\begin{cases} \phi_\Lambda(v) = e^{-\lambda}(P\phi_\Lambda)(v), & v \in V\setminus L,\\ \phi_\Lambda(i, m_i) = \gamma_i, & l_i \in \Lambda,\\ \phi_\Lambda(i, m_i) = e^{-\lambda}\phi_\Lambda(i, m_i - 1), & l_i \in \Lambda^c.\end{cases}$$

Then, we have

$$\phi_\Lambda(0) = \left(\sum_{k\in[\Lambda]} \frac{2\gamma_k}{\mu_+^{m_k} - \mu_-^{m_k}}\right) \times \Big(\sum_{k\in[\Lambda]} \frac{\mu_+^{m_k} + \mu_-^{m_k}}{\mu_+^{m_k} - \mu_-^{m_k}} + \sum_{k\in[\Lambda]^c} \frac{\mu_+^{m_k} - \mu_-^{m_k}}{\mu_+^{m_k} + \mu_-^{m_k}}\Big)^{-1}$$
$$= \left(\sum_{k\in[\Lambda]} \frac{\gamma_k}{\sinh m_k\kappa}\right) \times \Big(\sum_{k\in[\Lambda]} \coth m_k\kappa + \sum_{k\in[\Lambda]^c} \tanh m_k\kappa\Big)^{-1},$$

where $\mu_\pm = \mu_\pm(\lambda) = e^\lambda \pm \sqrt{e^{2\lambda} - 1}$ *and* $\kappa = \log \mu_+$.

Proof. We may assume that $\phi_\Lambda(k,x) = A_k\mu_+^x + B_k\mu_-^x$ for some A_k and B_k on each path, and from the boundary conditions at 0 and leaves, we obtain

$$\begin{cases} A_k + B_k = \phi_\Lambda(0), & k \in [K] \\ A_k\mu_+^{m_k} + B_k\mu_-^{m_k} = \gamma_k, & l_k \in \Lambda, \\ A_k\mu_+^{m_k} = B_k\mu_-^{m_k}, & l_k \in \Lambda^c, \end{cases}$$

For the third conditions, we used the equalities $\mu_+\mu_- = 1$ and $1 - e^{-\lambda}\mu_\pm = \mp\sqrt{1 - e^{-2\lambda}}$. Also we have

$$\sum_{k=1}^K A_k = \sum_{k=1}^K B_k$$

from $\phi_\Lambda(0) = e^{-\lambda}\frac{1}{K}\sum_{k\in[K]}\phi_\Lambda(k,1)$. By solving these equations, we obtain the assertion. □

Proof of Theorem 3.1. From (1.2), we see that

$$\mathbb{E}_0[e^{-\lambda C}] = \sum_{\substack{\Lambda\subset L \\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|+1}\mathbb{E}_0[e^{-\lambda\sigma_\Lambda}].$$

We set $\phi_\Lambda(v) = \mathbb{E}_v[e^{-\lambda\sigma_\Lambda}]$. Since ϕ_Λ satisfies the assumption in Lemma 3.13 with $\gamma_k = 1$, we obtain $\phi_\Lambda(0)$ as in Lemma 3.13. By setting

$$\nu(\{k\}) = (\sinh m_k\kappa)^{-1},$$
$$\mu_2(\{k\}) = \mu_1(\{k\})^{-1} = \tanh m_k\kappa,$$

in the second formula in Lemma 3.5 we obtain the desired integral representation. □

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Yusuke Higuchi
Mathematics Laboratories,
College of Arts and Sciences, Showa University,
4562 Kamiyoshida, Fujiyoshida, Yamanashi 403-0005, Japan.
E-mail: higuchi(at)cas.showa-u.ac.jp

Takuya Ohwa
Faculty of Mathematics, Kyushu University,
744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
E-mail: ohwa(at)math.kyushu-u.ac.jp

Tomoyuki Shirai
Faculty of Mathematics, Kyushu University,
744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
E-mail: shirai(at)math.kyushu-u.ac.jp