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Overview to mathematical analysis for fractional diffusion equations – new mathematical aspects motivated by industrial collaboration

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Abstract. The mathematics turns out to be useful for creation of innovations in the industry, and the mathematical knowledge and thinking manners are used effectively for that purpose. However, this is only one aspect of the industrial mathematics where various existing mathematical knowledge are applied for solving required subjects from industry. On the other hand, one can see the opposite direction; Pursuit of industrial purposes inspires to create new fields of mathematics by motivating and activating existing researches. is an important aspect of the industrial mathematics because it does not only give tools for solving concrete problems, but also enriches the existing branches of mathematics. In this article, as such a possible example, we discuss a fractional diffusion equation which has been studied already comprehensively from the theoretical interests, but the researches are expanded as a mathematical topic in view of the industrial applications.

Keywords. mathematics motivated by industrial mathematics, fractional diffusion equation, fractional calculus, well-posedness, qualitative properties

1. Introduction

The diffusion of contaminants under the ground is important and from the environmental viewpoint, better simulations and predictions of the density of the contaminant over time should be done. Morever the real size is over a few kilometers, while one can execute only laboratory experiments with meter sizes (see Figure 1).

As classical model equation, one can use a diffusion convection equation:

$$\rho(x)\frac{\partial u}{\partial t}(x,t) = div(p(x)\nabla u(x,t)) + b(x)\cdot\nabla u(x,t),$$

where $u(x,t)$ denotes the density at time t and the location x. In 1992, Adams and Gelhar [1] pointed that field data show anomalous diffusion in heterogeneous aquifer which can not be intrepreted by the classical convection-diffusion equation (see Figure 2). Since [1], there are trials for better modelling and we can refer to Berkowitz, Scher and Silliman [6], Y. Hatano and N. Hatano [18]. See also Berkowitz, Cortis, Dentz and Scher [5], Xiaong, G. Huang and Q. Huang [52]. The diffusion is observed to be slower than the prediction on the basis of the classical convection-diffusion equation, and such anomalous diffusion is called "slow diffusion". We refer especially to Y. Hatano and N. Hatano [18] where the continuous-time random walk is discussed. In the soil, one has to take into consideration the porosity and the heterogeneity of the medium, and by the microscopic level, one can conclude that the classical random walk model may not be suitable in view of the heterogeneity. The continuous-time random walk is a microscopic model for the anomalous diffusion, and by an argument similar to the derivation of the classical diffusion equation from the random walk, one can derive fractional diffusion models (e.g., Metzler and Klafter [34], (pp.14-18), Sokolov, Klafter and Blumen [50]).

The fractional diffusion equation can be described as follows. Let $0 < \alpha < 1$ throughout this paper. We consider

$$\partial_t^\alpha u(x,t) = (Lu)(x,t) + F(x,t), \quad x\in\Omega,\ t\in(0,T), \quad (1.1)$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, ∂_t^α denotes the Caputo fractional derivative with respect to t and is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u}{\partial \tau}(x,\tau)d\tau$$

for x-dependent function $u(x,t)$ and

$$D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{dg}{d\tau}(\tau)d\tau$$

for x-independent function $g(t)$ (e.g., Podlubny [41]), Γ is the Gamma function and the operator L is a symmetric uniformly ellptic operator:

$$(Lu)(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x), \quad x\in\Omega,$$

where $a_{ij} = a_{ji}, \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$, ≤ 0 on $\overline{\Omega}$ and we assume that there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j \ge \mu\sum_{i=1}^n \zeta_i^2$$

for all $x \in \overline{\Omega}$ and $\zeta_1, ..., \zeta_n \in R$. Moreover F is a given function in $\Omega \times [0,T]$ and $T > 0$ is a fixed value.

The fractional diffusion equation needs independent mathematical researches, even though one can discuss similarly to the classical convection-diffusion equation. One has to take into consideration that some properties for the natural number order derivatives fail for fractional order derivatives: For example, the derivative of the product of two functions and the sequential derivative do not hold.

We note that

$$\lim_{\alpha \to 1} D_t^\alpha g(t) = \frac{dg}{dt}(t), \quad 0 \le t \le T$$

for $g \in C^2[0,T]$. In fact, the integration by parts yields

$$D_t^\alpha g(t) = \frac{1}{\Gamma(2-\alpha)} \left(g'(0)t^{1-\alpha} + \int_0^t (t-s)^{1-\alpha} g''(s) ds \right) \longrightarrow g'(t)$$

as $\alpha \to 1$ for arbitrary $t \in [0,T]$.

This means that the Caputo derivative of order $\alpha \in (0,1)$ has an extended sense of the first-order derivative.

As theoretical backgrounds for e.g., better simulation requested for the environmental or possible industrial applications, one can apply mathematical results which have been already gained. However, in view of the applications, further mathematical researches may be necessary, which is quite a strong motivation for mathematicians and may open new aspects of the vast field of the fractional differential equation. That is, for better applications, mathematicians should sometimes modify the existing theories and even create and develop new branches in mathematics. This is bilaterally meaningful collaboration between mathematics and industry. We expect that the fractional diffusion equation may be such a topic. In this article, we intend a compact overview to such aspects concerning the fractional differential equations and present results which have been proved by the authors' group and their colleagues. As for more complete descriptions and the proofs, we refer to the original papers, e.g., Cheng, Nakagawa, Yamamoto and Yamazaki [7], Sakamoto [45] and Sakamoto and Yamamoto [46], and we omit.

The article is composed of 5 sections. In section 2, we discuss some specific aspects of fractional calculus and in section 3, we choose topics on ordinary fractional differential equations. In section 4, we will present results on the well-posedness of initial/boundary value problems for fractional diffusion equations to show qualitative properties which interpret the character as slow diffusion and in section 5 we discuss more properties related with inverse problems.

2. Fractional Calculus

For a function $g \in C^1[0,T]$, we recall

$$D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} g(\tau) d\tau$$

for $0 < \alpha < 1$. By the definition, for example, we can calculate:

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0. \tag{2.1}$$

The Caputo derivative is defined by the integral and so is not a local operation, and several properties for the usual calculus do not hold.

First we note that we have no useful formula for the derivative of product of two functions:

$$D_t^\alpha (fg) \ne (D_t^\alpha f)g + f D_t^\alpha g,$$

and accordingly we have no useful formula for the integration by parts (e.g., [41]).

Moreover we have no usual properties for sequential derivatives in general:

$$D_t^\alpha D_t^\beta \ne D_t^{\alpha+\beta}$$

even if $0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$.
In fact, let $0 < \alpha < \frac{1}{2}$. By (2.1), we have $D_t^\alpha t^\alpha = \Gamma(\alpha+1)$, and $D_t^\alpha (D_t^\alpha t^\alpha) = 0$, but

$$D_t^{2\alpha} t^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} t^{-\alpha},$$

that is, $D_t^\alpha (D_t^\alpha t^\alpha) \ne D_t^{2\alpha} t^\alpha$. On the other hand, we note by (2.1) that

$$D_t^\beta (D_t^\alpha t^\gamma) = D_t^{\alpha+\beta} t^\gamma$$

if $\gamma - \alpha > 0$. More generally we can prove

Proposition 2.1
Let $f \in C^2[0,T]$ and let $0 < \alpha, \beta < 1$, $\alpha + \beta < 1$. Then

$$D_t^\beta (D_t^\alpha f)(t) = D_t^{\alpha+\beta} f(t), \quad 0 \le t \le T.$$

Proof. We have

$$\begin{aligned} D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (f'(s) - f'(t)) ds \\ &\quad + \frac{1}{(1-\alpha)\Gamma(1-\alpha)} t^{1-\alpha} f'(t). \end{aligned}$$

Then

$$\begin{aligned} (D_t^\alpha f)'(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \partial_t ((t-s)^{-\alpha}) \\ &\quad \times (f'(s) - f'(t)) ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \Bigg(- \int_0^t (t-s)^{-\alpha} f''(t) ds \\ &\qquad + \frac{1}{1-\alpha} f''(t) t^{1-\alpha} + t^{-\alpha} f'(t) \Bigg). \end{aligned}$$

Since $\partial_t((t-s)^{-\alpha}) = -\partial_s((t-s)^{-\alpha})$, we have

$$\begin{aligned}
&\int_0^t \partial_t((t-s)^{-\alpha})(f'(s)-f'(t))ds \\
&= -\int_0^t \partial_s((t-s)^{-\alpha})(f'(s)-f'(t))ds \\
&= -\left[(t-s)^{-\alpha}(f'(s)-f'(t))\right]_{s=0}^{s=t} \\
&\quad + \int_0^t (t-s)^{-\alpha} f''(s)ds \\
&= t^{-\alpha}(f'(0)-f'(t)) \\
&\quad + \int_0^t (t-s)^{-\alpha} f''(s)ds.
\end{aligned}$$

Hence we have

$$(D_t^\alpha f)'(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha} f''(s)ds + t^{-\alpha}f'(0)\right).$$

Therefore

$$\begin{aligned}
(D_t^\beta D_t^\alpha f)(t) &= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(D_s^\alpha f(s))' ds \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left(\int_0^t (t-s)^{-\beta}\right. \\
&\quad \times \left(\int_0^s (s-\xi)^{-\alpha} f''(\xi)d\xi\right)ds \\
&\quad \left. + \int_0^t s^{-\alpha}(t-s)^{-\beta} ds f'(0)\right).
\end{aligned}$$

Noting $\int_0^t \left(\int_0^s d\xi\right) ds = \int_0^t \left(\int_\xi^t ds\right) d\xi$ and

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^t \left(\int_\xi^t (t-s)^{-\beta}\right. \\
&\qquad \left. \times (s-\xi)^{-\alpha} ds\right) f''(\xi)d\xi \\
&= \frac{1}{\Gamma(2-\alpha-\beta)}\int_0^t (t-\xi)^{1-\alpha-\beta} f''(\xi)d\xi,
\end{aligned}$$

by integration by parts, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^t \left(\int_\xi^t (t-s)^{-\beta}\right. \\
&\qquad \left. \times (s-\xi)^{-\alpha} ds\right) f''(\xi)d\xi \\
&= \frac{1}{\Gamma(2-\alpha-\beta)}\left\{\left[f'(\xi)(t-\xi)^{1-\alpha-\beta}\right]_{\xi=0}^{\xi=t}\right. \\
&\quad \left. + (1-\alpha-\beta)\int_0^t (t-\xi)^{-\alpha-\beta} f'(\xi)d\xi\right\},
\end{aligned}$$

that is, $(D_t^\beta D_t^\alpha f)(t) = D_t^{\alpha+\beta} f(t)$. The roof of the proposition is completed.

Moreover in Luchko [25], the following is proved.

Proposition 2.2
Let $g \in C^1[0,T]$ attain the maximum at $t = t_0 \in (0,T]$. Then

$$(D_t^\alpha g)(t_0) \ge 0.$$

On the other hand, we can not determine the local behaviour of g near $t = t_0$ by $D_t^\alpha g(t_0)$ because D_t^α is not a local operation.

As for further detailed account of fractional calculus, see Kilbas, Srivastava and Trujillo [20], Miller and Ross [35], Oldham and Spanier [38], Podlubny [41], Samko, Kilbas and Marichev [47].

3. Ordinary fractional differential equations

For discussions about the fractional diffusion equation, the ordinary fractional differential equation is useful and is an independent important topic. We consider

$$D_t^\alpha u(t) = F(u,t), \quad t > 0, \quad u(0) = a. \tag{3.1}$$

Here $a \in \mathbb{R}$ and F is a given function. First let $F(u,t) = \lambda u + f(t)$, where λ is a constant:

$$\begin{aligned}
D_t^\alpha(t) &= \lambda u(t) + f(t), \quad t > 0, \\
u(0) &= a.
\end{aligned} \tag{3.2}$$

In Gorenflo and Mainardi [15], Gorenflo and Rutman [17] (also see pp.140-141 in [20]), it is proved that there exists a unique solution to (3.2) and

$$\begin{aligned}
u(t) &= aE_{\alpha,1}(\lambda t^\alpha) \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)f(s)ds
\end{aligned} \tag{3.3}$$

for $t > 0$. Here $E_{\alpha,\beta}(t)$, $\alpha,\beta > 0$, is the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}$$

(see e.g., [41]) and is an entire function. Noting that

$$E_{1,1}(z) = e^z,$$

we see that (3.3) is $u(t) = ae^{\lambda t}$ for $\alpha = 1$ and $f \equiv 0$.

As for the unique existence of solution to (3.1), we can modify arguments in Chapter 3 in Kilbas, Srivastava and Trujillo [20] for example. See also Gorenflo and Mainardi [16].

Similarly to the ordinary differential equation, we can discuss the asymptotic behaviour and the dynamical system for the fractional differential equation. Here we will mention only few topics which should be exploited more.

Let us consider

$$D_t^\alpha U(t) = AU(t), \quad t > 0, \tag{3.4}$$

where $U = (u_1, ..., u_N)^T$, $\cdot^T$ denotes the transpose of the vector under consideration, and A is an $N \times N$ constant matrix. Then we can prove

Proposition 3.1

Let $0 < \alpha < 1$ and let the real parts of all the eigenvalues of A be negative. Then there exists a constant $C > 0$ such that

$$\|u(t)\| \le \frac{C}{t^\alpha}\|u(0)\|, \quad t > 0$$

for an arbitrary solution to (3.4).

Unlike the case $\alpha = 1$, we can not have the exponential decay. Moreover the decay rate $t^{-\alpha}$ is the best possible as the following example shows: Let $N = 1$ and consider (3.2) with $\lambda < 0$. Then by Theorem 1.4 (pp.33-34) in [41] implies that the solution can not decay faster than $t^{-\alpha}$. In sections 4 and 5, we discuss similar properties of solutions of the fractional diffusion equation as $t \to \infty$.

In view of Proposition 3.1, we can discuss the linearized stability for

$$D_t^\alpha u(t) = Au + F(u, t)$$

with vector-valued function u and a suitable nonlinear term F.

As other interesting problem, we can mention the global existence in time to a nonlinear ordinary fractional differential equation. For example let us consider:

$$D_t^\alpha u(t) = -u(1 - u), \quad t > 0. \tag{3.5}$$

In the case of $\alpha = 1$, the following is well-known and can be proved easily.

$0 < u(0) < 1$: the solution exists globally.

$u(0) > 1$: the solution can not exist globally.

However for $0 < \alpha < 1$, such a result is not known. The difficulty comes from that $D_t^\alpha u(t)$ does not give information of $u(t)$ near t (the converse to Proposition 2.2 is not true).

We further mention a few works on the chaos for systems of ordinary fractional differential equations and refer to Ge and Hsu [11], Li and Peng [23] where chaoses are observed by numerical simulations for some systems. In the latter paper, the authors consider

$$\begin{cases} D_t^{\alpha_1} u = a(v - u), \\ D_t^{\alpha_2} v = (c - a)u - uw + cv, \\ D_t^{\alpha_3} w = uv - bw, \end{cases}$$

where $0 < \alpha_1, \alpha_2, \alpha_3 \le 1$ and $a, b, c \in \mathbb{R}$. However comprehensive researches are not yet done. See also Luchko, Rivero, Trujillo and Pilar Velasco [27] which considers an inverse problem of determining a memory function in ordinay fractional differential equations.

4. Fractional diffusion equation

We survey results on the fractional diffusion equation (1.1).

The fractional diffusion equation has been introduced in physics by Nigmatullin [37] to describe diffusions in media with fractal geometry. One can regard (1.1) as a macroscopic model derived from the continuous-time random walk. Metzler and Klafter [34] demonstrated that a fractional diffusion equation describes a non-Markovian diffusion process with a memory. See also Metzler, Glöckle and Nonnenmacher [32], Metzler and Klafter [33], Roman [43]. Roman and Alemany [44] investigated a continuous time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically. Ginoa, Cerbelli and Roman [13] presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. Here we refer to several works on the mathematical treatments for equation (1.1). Kochubei [21], [22] applied the semigroup theory in Banach spaces, and Eidelman and Kochubei [8] constructed the fundamental solution in R^n and proved the maximum principle for the Cauchy problem. See also Mainardi [28] - [31] and Schneider and Wyss [49]. Gejji and Jafari [12] solved a nonhomogeneous fractional diffusion-wave equation in a 1-dimensional bounded domain. Fujita [10] discussed an integrodifferential equation which interpolates the heat equation and the wave equation in an unbounded domain. Agarwal [3] solved a fractional diffusion equation using a finite sine transform technique and presented numerical results in a 1-dimensional bounded domain.

We will solve equation (1.1) satisfying the following initial-boundary value conditions:

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \in (0, T), \tag{5.1}$$

$$u(x, 0) = a(x), \quad x \in \Omega. \tag{5.2}$$

In spite of the importance, to the authors' best knowledge, there are not many works published concerning the unique existence of the solution to (1.1), (5.1) and (5.2) and the properties are remarkably different from the standard diffusion. In Prüss [42] (especially in Chapter I.3), one can refer to the methods for (1.1). See also Bazhlekova [4] and Gorenflo, Luchko and Zabrejko [14], Gorenflo and Mainardi [16].

The maximum principle for (1.1) with (5.1) is recently proved in Luchko [25] and see also a new paper Luchko [26] which proves the well-posedness of the forward problem (1.1), (5.1) and (5.2), but we will here present more detailed regularity and qualitative properties.

In particular, for discussions on inverse problems, we need representation formulae of the solution to (1.1), (5.1) and (5.2) by the eigenfunctions, and to the authors' best knowledge, there are no results published concerning the regularity properties of the eigenfunction expansions of the solutions which are corresponding to Chapter 3 of Lions and Magenes [24] and Pazy [39] for example.

In this section, we will show the well-posedness of the solution given by the Fourier method. Second we establish several uniqueness results for related inverse problems.

Let $L^2(\Omega)$ be a usual L^2-space with the scalar product $(\cdot, \cdot)$. We denote the Sobolev spaces by $H^\ell(\Omega)$ with $\ell > 0$ (e.g., Adams [2]).

We define an operator L in $L^2(\Omega)$ by

$$(Lu)(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x), \quad x \in \Omega,$$
$$\mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega).$$

Here and henceforth C_j denote positive constants which are independent of F in (1.1), a, b in (5.1) and (5.2), but may depend on α and the coefficients of the operator L.

Since $-L$ is a symmetric uniformly ellitpic operator, the spectrum of L is entirely composed of eigenvalues and counting according to the multiplicities, we can set: $0 < \lambda_1 \le \lambda_2 \le \cdots$. By $\varphi_n \in H^2(\Omega)\cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $-\lambda_n$: $L\varphi_n = -\lambda_n\varphi_n$. Then the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is orthonormal basis in $L^2(\Omega)$.

We are ready to state our main theorems on the unique existence of solution to (1.1), (5.1) and (5.2).

Theorem 4.1
Let $F = 0$.
(i) For $a \in L^2(\Omega)$, there exists a unique solution $u \in C([0,T];L^2(\Omega))\cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ such that $\partial_t^\alpha u \in C((0,T];L^2(\Omega))$. Moreover there exists a constant $C_1 > 0$ such that

$$\|u\|_{C([0,T];L^2(\Omega))} \le C_1\|a\|_{L^2(\Omega)},$$
$$\|u(\cdot,t)\|_{H^2(\Omega)} + \|\partial_t^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C_1 t^{-\alpha}\|a\|_{L^2(\Omega)}$$

for all $a \in L^2(\Omega)$. The eigenfunction expansion holds:

$$u(x,t) = \sum_{n=1}^{\infty}(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)$$

in $C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$.
(ii) There exists a constant $C_2 > 0$ such that

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(\Omega\times(0,T))} \le C_2\|a\|_{H^1(\Omega)}$$

for all $a \in H_0^1(\Omega)$.
(iii) There exists a constant $C_3 > 0$ such that

$$\|u\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C_3\|a\|_{H^2(\Omega)}$$

for $a \in H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 4.2
Let $a = 0$ and $F \in L^2(\Omega\times(0,T))$. Then there exists a unique solution $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and there exists a constant $C_4 > 0$ such that

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(\Omega\times(0,T))} \le C_4\|F\|_{L^2(\Omega\times(0,T))}$$

for all $F \in L^2(\Omega\times(0,T))$. Moreover

$$u(x,t) = \sum_{n=1}^{\infty}\int_0^t (F(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1} \times E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\varphi_n(x).$$

Remark. These resutls include the case of $\alpha = 1$

For $\theta \in (0,1)$, we set

$$\|F\|_{C^\theta([0,T];L^2(\Omega))} = \|F\|_{C([0,T];L^2(\Omega))} + \sup_{0\le t<s\le T}\frac{\|F(\cdot,t)-F(\cdot,s)\|_{L^2(\Omega)}}{|t-s|^\theta}.$$

Next we show the maximal regularity for $F \in C^\theta([0,T];L^2(\Omega))$.

Theorem 4.3
Let $a \in L^2(\Omega)$ and $F \in C^\theta([0,T];L^2(\Omega))$.
The solution u is represented by eigenfunction expansion:

$$u(x,t) = \sum_{n=1}^{\infty}\left\{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) + \int_0^t (F(\cdot,t-\tau),\varphi_n)\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)d\tau\right\}\varphi_n(x).$$

(1) For arbitrary $\delta > 0$, there exists a constant $C_5 = C_5(\delta) > 0$ such that

$$\|Lu\|_{C^\theta([\delta,T];L^2(\Omega))} + \|\partial_t^\alpha u\|_{C^\theta([\delta,T];L^2(\Omega))} \le \frac{C_5}{\delta}(\|F\|_{C^\theta([\delta,T];L^2(\Omega))} + \|a\|_{L^2(\Omega)})$$

for all $a \in L^2(\Omega)$ and $F \in C^\theta([0,T];L^2(\Omega))$.
(2) There exists a constant $C_6 > 0$ such that

$$\|Lu\|_{C([0,T];L^2(\Omega))} + \|\partial_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C_9(\|a\|_{H^2(\Omega)} + \|F\|_{C^\theta([0,T];L^2(\Omega))})$$

for $a \in H^2(\Omega)\cap H_0^1(\Omega)$ and $F \in C^\theta([0,T];L^2(\Omega))$.
(3) Let $a = 0$. There exists a constant $C_7 > 0$ such that

$$\|Lu\|_{C^\theta([0,T];L^2(\Omega))} + \|\partial_t^\alpha u\|_{C^\theta([0,T];L^2(\Omega))} \le C_7\|F\|_{C^\theta([0,T];L^2(\Omega))}$$

for all $F \in C^\theta([0,T];L^2(\Omega))$ satisfying $F(\cdot,0) = 0$.

This is the same as the case of $\alpha = 1$. Prüss [42] already proved Theorem 4.3 (3).

Corollary 4.1 (slow decay)
Let $a \in L^2(\Omega)$ and $F = 0$. Then thers exist constants $C_8, C_9 > 0$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_8}{1+\lambda_1 t^\alpha}\|a\|_{L^2(\Omega)}, \quad t \ge 0,$$

$$\|\partial_t^m u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_9}{t^m}\|a\|_{L^2(\Omega)}, \quad t > 0,\ m \in \mathbb{N}$$

for all $a \in L^2(\Omega)$. Here C_9 is independent of m.

Here we compare our results with the case $\alpha = 1$: $t^{-\alpha}$-decay for $0 < \alpha < 1$ but the exponential decay for $\alpha = 1$.

We can consider $\alpha > 1$ similarly and see Sakamoto [45], Sakamoto and Yamamoto [46].

5. Further qualitative results for the fractional diffusion equation

5.1. Backward problem in time

It is well-known that the backward problem in time is severaly ill-posed for the parabolic problem (i.e., $\alpha = 1$). The severe ill-posedness means that we can not recover the stability in the backward problem even if we strengthen the norm within Sobolev norms for estmating the initial value in $L^2(\Omega)$. For $0 < \alpha < 1$, the backward problem is moderately ill-posed, as the following theorem imlies:

Theorem 5.1
Let $0 < \alpha < 1$. For arbitrary $T > 0$ and arbitrary $a_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ such that $u(\cdot, T) = a_1$ to the forward problem (1.1), (5.1) and (5.2) with $F = 0$. Moreover there exist constants $C_{10}, C_{11} > 0$ such that

$$C_{10}\|u(\cdot,0)\|_{L^2(\Omega)} \le \|u(\cdot,T)\|_{H^2(\Omega)}$$

$$\le C_{11}\|u(\cdot,0)\|_{L^2(\Omega)}.$$

5.2. Uniqueness of solution

The solution can be uniquely determined by data in any small subdomain over time interval. This is closely related with the approximate controllability (e.g., Georg Schmidt and Weck [48]) but we will omit further discussions.

Theorem 5.2
Let $0 < \alpha < 1$.
Let spatial dimension ≤ 3, $a \in H_0^4(\Omega)$,

$$\partial_t^\alpha u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}u) + c(x)u,$$

$u|_{\partial\Omega} = 0$ and $u = 0$ in $\omega \times (0,T)$ with arbitrary subdomain ω and $T > 0$. Then $u = 0$ in $\Omega \times (0,T)$.

5.3. Decay at $t = \infty$

Non-trivial solutions can not decay faster than polynomial orders, which implies the slow diffusion for $0 < \alpha < 1$. See also Corollary 4.1 in section 4.

Theorem 5.3 *Let $0 < \alpha < 1$, ω be an arbitrary subdomain, let spatial dimension ≤ 3 and $a \in H_0^4(\Omega), \partial_t^\alpha u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}u) + c(x)u$, $u|_{\partial\Omega} = 0$. Let for all $m \in \mathbb{N}$, there exists a constant $C(m) > 0$ such that $\|u(\cdot,t)\|_{L^\infty(\omega)} \le \frac{C(m)}{t^m}$ as $t \to \infty$. Then $u = 0$ in $\Omega \times (0,\infty)$.*

5.4. Other inverse problem

Let $p > 0$ on $[0,\ell]$ and $p \in C^2([0,\ell])$. We consider

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right),\ 0 < x < \ell,\ 0 < t < T,$$

$$u(x,0) = \delta(x) : \text{delta function},$$

$$u_x(0,t) = u_x(\ell,t) = 0.$$

It is practically difficult to determine the order α a priori and it is important to determine the order α and the diffusion coefficient $p(x)$ by available observation data at the boundary point $x = 0$ over time interval. Thus the following inverse problem is significant.
Inverse problem: Determine $\alpha \in (0,1)$ and $p(x)$, $0 < x < \ell$ by $u(0,t)$, $0 < t < T$.

Then the uniqueness is proved in Cheng, Nakagawa, Yamamoto and Yamazaki [7] by means of the Gel'fanf-Levitan theory (see e.g., Freiling and Yurko [9]) and the eigenfunction expansion. Fixed $\alpha = 1$, a similar inverse problem is considered in Murayama [36], Pierce [40], Suzuki and Murayama [51]. By the results in section 4, we can consider other types of inverse problems and we refer to Isakov [19] as monographs on inverse problems for partial differential equations.

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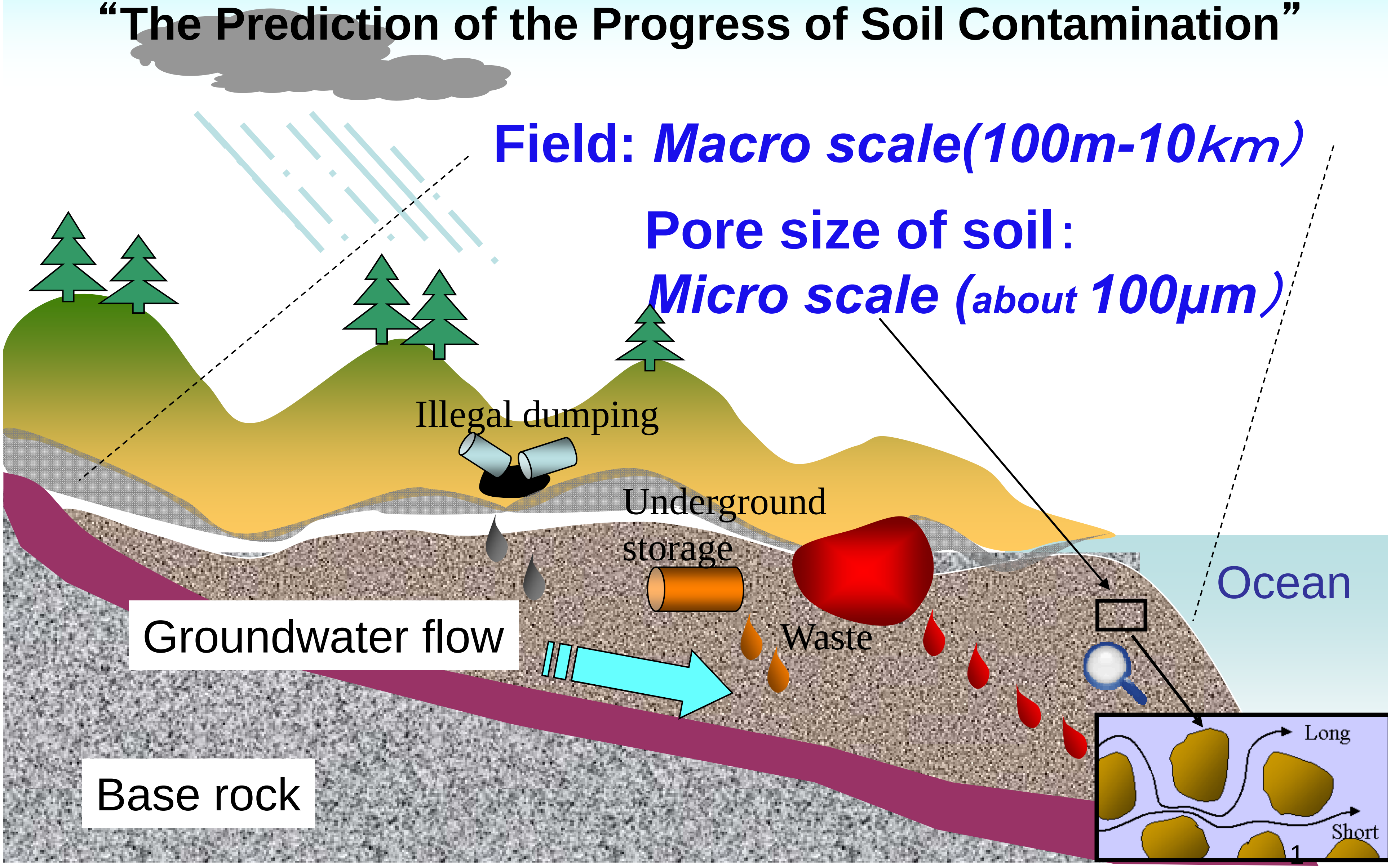

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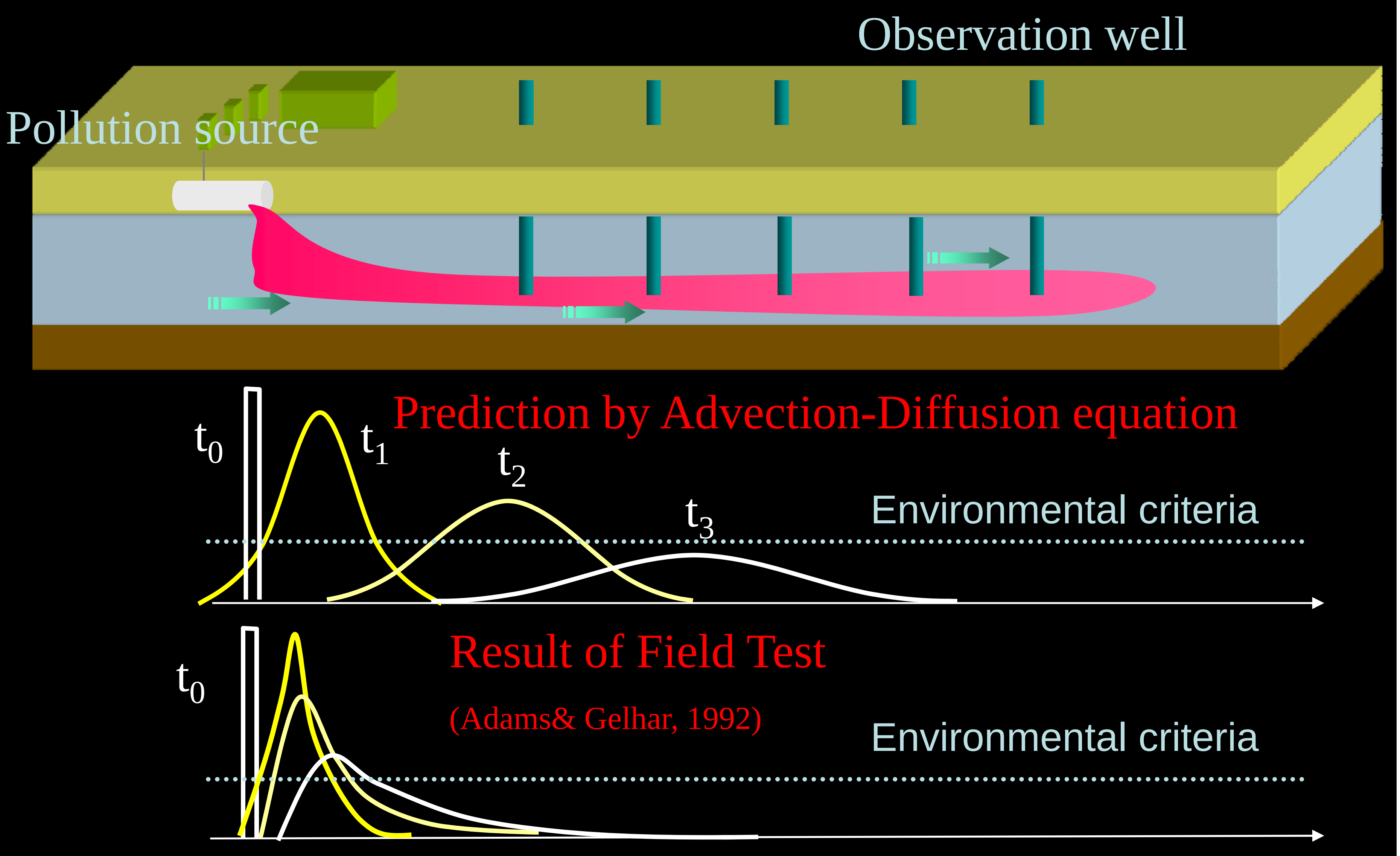

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